

TD 2 — Questions and Solutions

Lectures 2 and 3 · Each question with its step-by-step solution



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Operations Research (O.R.)

Goal of this session: each problem is solved twice. **A:** with a picture (graphical method and corner method, Lecture 2). **B:** with the simplex method (Lecture 3). Then we compare: the simplex method walks along the corners of the picture and stops at the same optimal plan.

Rules. Draw every figure with a ruler, and compute every corner *exactly* by solving two equations. In the simplex method: the entering variable has the most negative number in the z -line of the tableau (the largest positive coefficient in a dictionary); if two numbers are equal, take the smallest index. The leaving variable has the smallest ratio, computed only with the *positive* numbers of the entering column. Write every row operation, use fractions (not decimals), and check the final plan in the original constraints.

In class (90 min): Exercise 1, 35 min · Exercise 2, 25 min · Exercise 3, 30 min. **At home:** Exercise 4.

How to read this document. Each exercise starts with its statement in a frame; each question is in a shaded box, and its solution follows. Key facts, methods, rules and cases refer to the written lectures of Lectures 2 and 3. In the tableaux, the pivot is in a box, the entering number of the z -line and the smallest ratio are in bold. L_1, L_2, L_3 are the lines of the tableau, from top to bottom.

Solution 1 Control boards

Recall the model of TD 1, Exercise 2: x_1, x_2 = numbers of Basic and Pro boards produced per day (profit in thousand DA),

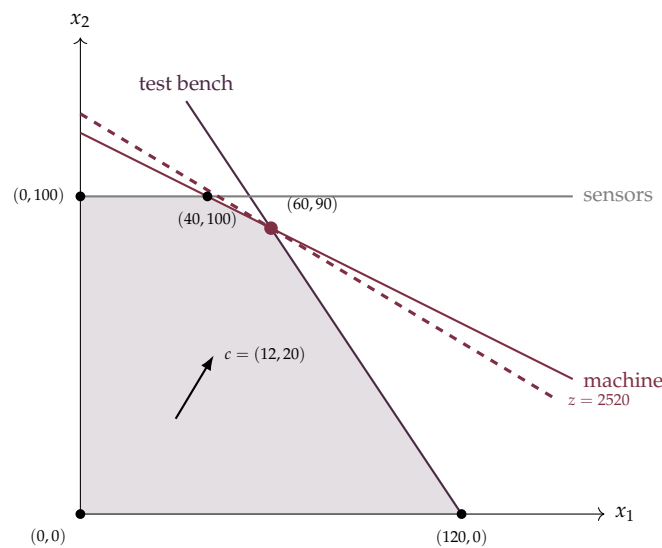
$$\max z = 12x_1 + 20x_2 \quad \text{s.t.} \quad \begin{cases} 2x_1 + 4x_2 \leq 480 & (\text{assembly machine, minutes}) \\ 3x_1 + 2x_2 \leq 360 & (\text{test bench, minutes}) \\ x_2 \leq 100 & (\text{special sensors}) \\ x_1, x_2 \geq 0. \end{cases}$$

A. With a picture

1 Draw the feasible set and compute the coordinates of all its corners.

The lines are $2x_1 + 4x_2 = 480$ (through $(240, 0)$ and $(0, 120)$), $3x_1 + 2x_2 = 360$ (through $(120, 0)$ and $(0, 180)$) and $x_2 = 100$ (horizontal). The origin satisfies the three constraints, so each time we keep the side of the origin. The corners are:

- ▶ $(0, 0)$; $(120, 0)$ (test bench line and $x_2 = 0$); $(0, 100)$ (sensor line and $x_1 = 0$);
- ▶ $(40, 100)$: $x_2 = 100$ and $2x_1 + 400 = 480$, so $x_1 = 40$ (check the test bench: $120 + 200 = 320 \leq 360$);
- ▶ $(60, 90)$: solve $2x_1 + 4x_2 = 480$ and $3x_1 + 2x_2 = 360$. The second gives $2x_2 = 360 - 3x_1$; put it in the first: $2x_1 + 2(360 - 3x_1) = 480$, so $-4x_1 = -240$, $x_1 = 60$, and $2x_2 = 360 - 180 = 180$, $x_2 = 90$.



2 Solve the problem with the graphical method, and check with the corner method. Give the optimal plan, the optimal value, the binding constraints, and the number of special sensors not used.

Graphical method. The lines $12x_1 + 20x_2 = k$ are parallel; k grows in the direction of the arrow $c = (12, 20)$. Pushing the line as far as possible, the last point touched is $(60, 90)$ (dashed line on the figure).

Corner method.

Corner	$(0, 0)$	$(120, 0)$	$(60, 90)$	$(40, 100)$	$(0, 100)$
$z = 12x_1 + 20x_2$	0	1440	$720 + 1800 = \mathbf{2520}$	$480 + 2000 = 2480$	2000

The best corner is $(60, 90)$: the same answer. The optimal plan is **60 Basic and 90 Pro boards per day**, with $z^* = 2520$ thousand DA.

Binding constraints. At $(60, 90)$: machine $120 + 360 = 480$ (binding), test bench $180 + 180 = 360$ (binding), sensors $90 < 100$ (not binding): **10 special sensors are not used.**

B. With the simplex method

3 Write the first dictionary, then run the simplex method with dictionaries (Method 3.1). At each step write: who enters, the ratio test, who leaves, the new dictionary, and the new corner with its value of z .

First dictionary (basic: s_1, s_2, s_3 ; non-basic: x_1, x_2):

$$s_1 = 480 - 2x_1 - 4x_2, \quad s_2 = 360 - 3x_1 - 2x_2, \quad s_3 = 100 - x_2, \quad z = 0 + 12x_1 + 20x_2.$$

With $x_1 = x_2 = 0$ we are at the corner $(0, 0)$, with $z = 0$.

Step 1. The largest positive coefficient is 20, so x_2 **enters**. Ratio test ($x_1 = 0, x_2 = t$): $s_1 = 480 - 4t$ gives $t \leq 120$; $s_2 = 360 - 2t$ gives $t \leq 180$; $s_3 = 100 - t$ gives $t \leq 100$. The smallest is 100, so s_3 **leaves**. Pivot: $x_2 = 100 - s_3$, and

$$\begin{aligned} s_1 &= 480 - 2x_1 - 4(100 - s_3) = 80 - 2x_1 + 4s_3, \\ s_2 &= 360 - 3x_1 - 2(100 - s_3) = 160 - 3x_1 + 2s_3, \\ z &= 0 + 12x_1 + 20(100 - s_3) = 2000 + 12x_1 - 20s_3. \end{aligned}$$

New corner $(0, 100)$, $z = 2000$.

Step 2. Only x_1 has a positive coefficient (12): x_1 **enters**. Ratio test ($s_3 = 0, x_1 = t$): $s_1 = 80 - 2t$ gives $t \leq 40$; $s_2 = 160 - 3t$ gives $t \leq \frac{160}{3} \approx 53.3$; $x_2 = 100 - s_3$ has no x_1 : no limit. The smallest is 40, so s_1 **leaves**. Pivot: $x_1 = 40 - \frac{1}{2}s_1 + 2s_3$, and

$$\begin{aligned} s_2 &= 160 - 3(40 - \frac{1}{2}s_1 + 2s_3) + 2s_3 = 40 + \frac{3}{2}s_1 - 4s_3, \\ z &= 2000 + 12(40 - \frac{1}{2}s_1 + 2s_3) - 20s_3 = 2480 - 6s_1 + 4s_3. \end{aligned}$$

New corner $(40, 100)$, $z = 2480$.

Step 3. s_3 has a positive coefficient (4): s_3 **enters**. Ratio test ($s_1 = 0, s_3 = t$): $x_1 = 40 + 2t$ grows: no limit; $s_2 = 40 - 4t$ gives $t \leq 10$; $x_2 = 100 - t$ gives $t \leq 100$. The smallest is 10, so s_2 **leaves**. Pivot: $4s_3 = 40 + \frac{3}{2}s_1 - s_2$, so $s_3 = 10 + \frac{3}{8}s_1 - \frac{1}{4}s_2$, and

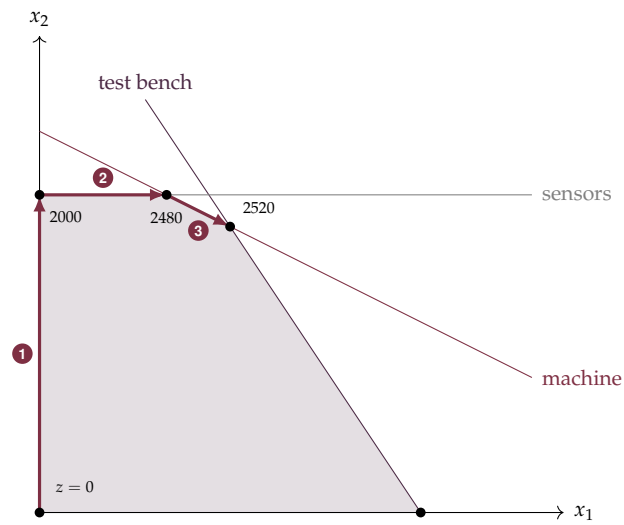
$$\begin{aligned} x_1 &= 40 - \frac{1}{2}s_1 + 2(10 + \frac{3}{8}s_1 - \frac{1}{4}s_2) = 60 + \frac{1}{4}s_1 - \frac{1}{2}s_2, \\ x_2 &= 100 - (10 + \frac{3}{8}s_1 - \frac{1}{4}s_2) = 90 - \frac{3}{8}s_1 + \frac{1}{4}s_2, \\ z &= 2480 - 6s_1 + 4(10 + \frac{3}{8}s_1 - \frac{1}{4}s_2) = 2520 - \frac{9}{2}s_1 - s_2. \end{aligned}$$

New corner $(60, 90)$, $z = 2520$, with $s_3 = 10$.

4 Explain with the last z -line why the last corner is optimal. On your figure of question 1, draw the path followed by the simplex method.

Optimality. The last z -line is $z = 2520 - \frac{9}{2}s_1 - s_2$: no positive coefficient. For every feasible plan $s_1, s_2 \geq 0$, so $z \leq 2520$; and $(60, 90)$ gives 2520. So it is optimal (Key fact 3.2): the same plan as in part A, and $s_3 = 10$ unused sensors, as in question 2. (The numbers $\frac{9}{2}$ and 1 give a proof without any picture: $\frac{9}{2} \times (2x_1 + 4x_2 \leq 480) + 1 \times (3x_1 + 2x_2 \leq 360)$ gives exactly $12x_1 + 20x_2 \leq 2160 + 360 = 2520$.)

Path. $(0,0) \rightarrow (0,100) \rightarrow (40,100) \rightarrow (60,90)$: up the x_2 -axis to the sensor line, right along the sensor line to the machine line, then down along the machine line to the test bench line. Three steps, and z grows each time: $0 \rightarrow 2000 \rightarrow 2480 \rightarrow 2520$.



Solution 2 Corners and basic solutions

$$\max z = 3x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 4, \quad x_1 + 3x_2 \leq 9, \quad x_1 \leq 3, \quad x_1, x_2 \geq 0.$$

A. With a picture

1 Draw the feasible set, compute its corners, and find the optimal plan with the corner method.

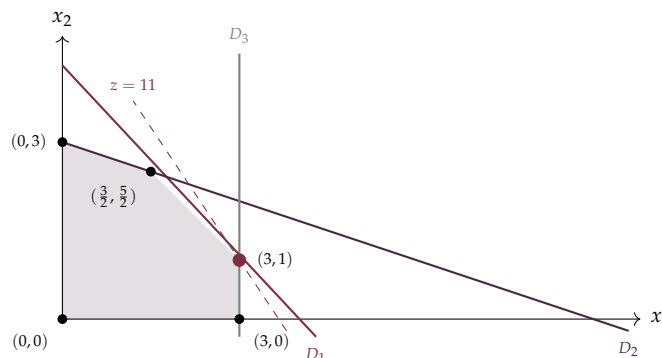
We call the lines $D_1: x_1 + x_2 = 4$, $D_2: x_1 + 3x_2 = 9$, $D_3: x_1 = 3$. The origin satisfies the three constraints, so we keep its side each time. The corners are:

- ▶ $(0,0)$; $(3,0)$ (D_3 and $x_2 = 0$; check $D_1: 3 \leq 4$);
- ▶ $(3,1)$ (D_3 and $D_1: x_2 = 4 - 3 = 1$; check $D_2: 3 + 3 = 6 \leq 9$);
- ▶ $(\frac{3}{2}, \frac{5}{2})$ (D_1 and D_2 : subtract them, $2x_2 = 5$, so $x_2 = \frac{5}{2}$ and $x_1 = \frac{3}{2}$; check $D_3: \frac{3}{2} \leq 3$);
- ▶ $(0,3)$ (D_2 and $x_1 = 0$: $3x_2 = 9$; check $D_1: 3 \leq 4$).

Other intersections are outside: for example D_1 meets $x_1 = 0$ at $(0,4)$, but $0 + 12 = 12 > 9$ breaks the second constraint.

Corner	$(0,0)$	$(3,0)$	$(3,1)$	$(\frac{3}{2}, \frac{5}{2})$	$(0,3)$
$z = 3x_1 + 2x_2$	0	9	11	$\frac{9}{2} + 5 = \frac{19}{2}$	6

The optimal plan is $(3,1)$, with $z^* = 11$. The line $3x_1 + 2x_2 = 11$ touches the polygon only at this corner.



B. With the simplex method

2 Write the standard form with slack variables s_1, s_2, s_3 . At each corner of question 1, give the values of the five variables. How many of them are zero? (These are the basic feasible solutions, Key fact 3.1.)

Standard form: $x_1 + x_2 + s_1 = 4$, $x_1 + 3x_2 + s_2 = 9$, $x_1 + s_3 = 3$, all five variables ≥ 0 . The slacks are $s_1 = 4 - x_1 - x_2$, $s_2 = 9 - x_1 - 3x_2$, $s_3 = 3 - x_1$:

Corner	x_1	x_2	s_1	s_2	s_3
$(0,0)$	0	0	4	9	3
$(3,0)$	3	0	1	6	0
$(3,1)$	3	1	0	3	0
$(\frac{3}{2}, \frac{5}{2})$	$\frac{3}{2}$	$\frac{5}{2}$	0	0	$\frac{3}{2}$
$(0,3)$	0	3	1	0	3

At each corner exactly **two** of the five variables are zero: one for each line through the corner ($s_1 = 0$ on D_1 , $s_2 = 0$ on D_2 , $s_3 = 0$ on D_3 , $x_1 = 0$ or $x_2 = 0$ on an axis). These two are the non-basic variables; the three others are the basic variables. So each corner is a basic feasible solution, as Key fact 3.1 says.

3 Solve the problem with the simplex method in tableaux (Method 3.2). Which corners are visited? Draw the path on your figure, and compare the result with question 1.

Pivot 1. The most negative number is -3 : x_1 enters. Ratios $4/1, 9/1, 3/1$: the smallest is 3, in L_3 ; s_3 leaves.

Tableau 0	x_1	x_2	s_1	s_2	s_3	RHS	ratio
s_1	1	1	1	0	0	4	4
s_2	1	3	0	1	0	9	9
s_3	1	0	0	0	1	3	3
z	-3	-2	0	0	0	0	

Row operations: L_3 unchanged (the pivot is 1); $L_1 \leftarrow L_1 - L_3$; $L_2 \leftarrow L_2 - L_3$; $z \leftarrow z + 3L_3$.

Pivot 2. The only negative number is -2 : x_2 enters. Ratios: $1/1 = 1, 6/3 = 2$; the line of x_1 has 0: no ratio. The smallest is 1, in L_1 ; s_1 leaves.

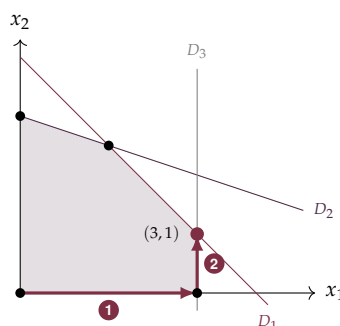
Tableau 1	x_1	x_2	s_1	s_2	s_3	RHS	ratio
s_1	0	1	1	0	-1	1	1
s_2	0	3	0	1	-1	6	2
x_1	1	0	0	0	1	3	—
z	0	-2	0	0	3	9	

Row operations: L_1 unchanged; $L_2 \leftarrow L_2 - 3L_1$; L_3 unchanged (0 in the column); $z \leftarrow z + 2L_1$.

Tableau 2	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	1	0	-1	1
s_2	0	0	-3	1	2	3
x_1	1	0	0	0	1	3
z	0	0	2	0	1	11

Answer. No negative number in the z -line: optimal. $x_1 = 3, x_2 = 1, s_2 = 3, z = 11$: the same corner as in question 1. The z -line reads $z = 11 - 2s_1 - s_3$, so $z \leq 11$ for every feasible plan.

Path. The tableaux visit $(0, 0)$ ($z = 0$), then $(3, 0)$ ($z = 9$), then $(3, 1)$ ($z = 11$): right along the x_1 -axis to D_3 , then up D_3 to D_1 . Only 3 of the 5 corners, and 2 pivots.

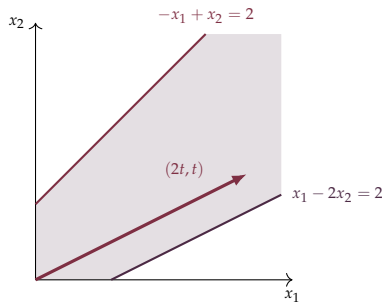


Solution 3 Two special cases, seen twice

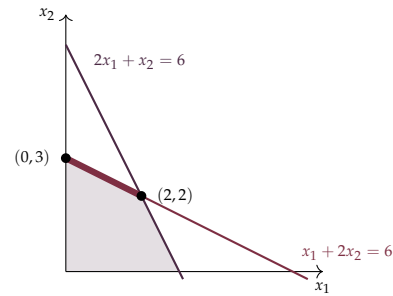
- (a) $\max z = x_1 + x_2$ s.t. $-x_1 + x_2 \leq 2$, $x_1 - 2x_2 \leq 2$, $x_1, x_2 \geq 0$.
 (b) $\max z = 2x_1 + 4x_2$ s.t. $x_1 + 2x_2 \leq 6$, $2x_1 + x_2 \leq 6$, $x_1, x_2 \geq 0$.

A. With a picture

1 For each problem, draw the feasible set and say which case it is: unique optimum, infinitely many optima, unbounded problem, or infeasible problem. Justify your answer.



(a): the set goes to infinity



(b): the thick edge is optimal

(a) Unbounded problem. The feasible set goes to infinity, between the two lines. The points $(2t, t)$, $t \geq 0$, are feasible: $-2t + t = -t \leq 2$, $2t - 2t = 0 \leq 2$, and $2t, t \geq 0$. Their value is $z = 2t + t = 3t$, as large as we want. There is no optimal plan.

(b) Infinitely many optima. The corners are $(0,0)$, $(3,0)$, $(2,2)$, $(0,3)$, where $(2,2)$ solves $x_1 + 2x_2 = 6$ and $2x_1 + x_2 = 6$ (add them: $3x_1 + 3x_2 = 12$, so $x_1 + x_2 = 4$; subtract this from the first: $x_2 = 2$, then $x_1 = 2$). The values of $2x_1 + 4x_2$ are 0, 6, 12, 12. The corners $(2,2)$ and $(0,3)$ tie: the objective $2(x_1 + 2x_2)$ is parallel to the first constraint. All the points of the edge between them are optimal: $(2 - 2t, 2 + t)$, $0 \leq t \leq 1$, with $z^* = 12$.

B. With the simplex method

2 Solve (a) with tableaux. Which signal of Lecture 3 appears? Give a family of feasible plans $x(t)$, $t \geq 0$, along which $z \rightarrow +\infty$, and compare with your picture.

The z -line has -1 and -1 : a tie, so we take x_1 (smallest index). Only L_2 has a positive number in the column of x_1 : ratio $2/1 = 2$; s_2 leaves.

Tableau 0	x_1	x_2	s_1	s_2	RHS	ratio
s_1	-1	1	1	0	2	—
s_2	1	-2	0	1	2	2
z	-1	-1	0	0	0	

Row operations: L_2 unchanged (the pivot is 1); $L_1 \leftarrow L_1 + L_2$; $z \leftarrow z + L_2$.

Tableau 1	x_1	x_2	s_1	s_2	RHS	ratio
s_1	0	-1	1	1	4	—
x_1	1	-2	0	1	2	—
z	0	-3	0	1	2	

Signal: no ratio. Now x_2 should enter (-3), but its column has only negative numbers (-1 and -2): there is no ratio. The problem is **unbounded** (Rule 3.3). With $x_2 = t$ and $s_2 = 0$, the tableau gives $x_1 = 2 + 2t$, $s_1 = 4 + t$ and $z = 2 + 3t$: the plans $x(t) = (2 + 2t, t)$, $t \geq 0$, are feasible and $z \rightarrow +\infty$. For example, $t = 10$ gives the feasible plan $(22, 10)$ with $z = 32$ (check: $-22 + 10 = -12 \leq 2$ and $22 - 20 = 2 \leq 2$).

Comparison. On the picture, the simplex method went from $(0, 0)$ to the corner $(2, 0)$, then found that it can go to infinity along the line $x_1 - 2x_2 = 2$, in the direction $(2, 1)$: the same direction as the family $(2t, t)$ of part A.

3 Solve (b) with tableaux. Which signal of Lecture 3 appears? Do one more pivot to find a second optimal corner, give all the optimal plans, and compare with your picture.

The most negative number is -4 : x_2 enters. Ratios: $6/2 = 3$, $6/1 = 6$: s_1 leaves.

Tableau 0	x_1	x_2	s_1	s_2	RHS	ratio
s_1	1	2	1	0	6	3
s_2	2	1	0	1	6	6
z	-2	-4	0	0	0	

Row operations: $L_1 \leftarrow L_1/2$; $L_2 \leftarrow L_2 - L_1$ (with the new L_1); $z \leftarrow z + 4L_1$.

Tableau 1	x_1	x_2	s_1	s_2	RHS
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	3
s_2	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	3
z	0	0	2	0	12

Signal: a 0 in the z-line. No negative number: the corner $(0, 3)$ is optimal with $z = 12$. But the non-basic variable x_1 has a 0 in the z-line (Rule 3.4). One more pivot with x_1 entering: ratios $3/\frac{1}{2} = 6$ and $3/\frac{3}{2} = 2$, so s_2 leaves. Row operations: $L_2 \leftarrow \frac{2}{3}L_2$; $L_1 \leftarrow L_1 - \frac{1}{2}L_2$; z unchanged (0 in the column).

Tableau 2	x_1	x_2	s_1	s_2	RHS
x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	2
x_1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	2
z	0	0	2	0	12

All the optimal plans. The corner $(2, 2)$ also gives $z = 12$. All the points of the edge between $(0, 3)$ and $(2, 2)$ are optimal, with $z^* = 12$: exactly the thick edge of the picture of part A.

Solution 4 At home — The drone workshop, by another path

Recall the drone workshop of the lectures: x_1, x_2 = numbers of drones A and B made per week (profit in thousand DA),

$$\max z = 40x_1 + 60x_2 \quad \text{s.t.} \quad \begin{cases} x_1 + 2x_2 \leq 40 & (\text{assembly hours}) \\ 3x_1 + 2x_2 \leq 60 & (\text{testing hours}) \\ x_2 \leq 18 & (\text{cameras}) \\ x_1, x_2 \geq 0. \end{cases}$$

A. With a picture

1 Draw the feasible set, compute its corners O, A, B, C, D, and find the optimal plan with the corner method.

The origin satisfies the three constraints, so we keep its side each time. The corners are:

- ▶ O = (0,0); A = (20,0) (testing line and $x_2 = 0$: $3x_1 = 60$; check assembly: $20 \leq 40$);
- ▶ B = (10,15) (assembly and testing lines: subtract them, $2x_1 = 20$, so $x_1 = 10$ and $2x_2 = 30$, $x_2 = 15$; check cameras: $15 \leq 18$);
- ▶ C = (4,18) (camera and assembly lines: $x_1 = 40 - 36 = 4$; check testing: $12 + 36 = 48 \leq 60$);
- ▶ D = (0,18) (camera line and $x_1 = 0$; check assembly: $36 \leq 40$).

Corner	O (0,0)	A (20,0)	B (10,15)	C (4,18)	D (0,18)
$z = 40x_1 + 60x_2$	0	800	$400 + 900 = 1300$	$160 + 1080 = 1240$	1080

The optimal plan is B: **10 drones A and 15 drones B per week**, for $z^* = 1300$ thousand DA.

2 Solve the problem with tableaux, but choose x_1 as the first entering variable (the lecture chose x_2). Which corners are visited? How many pivots are needed? Draw the path on your figure and compare with the path of the lecture.

Pivot 1. x_1 enters (our choice). Ratios $40/1 = 40$ and $60/3 = 20$ (the camera line has 0: no ratio): s_2 leaves.

Tableau 0	x_1	x_2	s_1	s_2	s_3	RHS	ratio
s_1	1	2	1	0	0	40	40
s_2	3	2	0	1	0	60	20
s_3	0	1	0	0	1	18	—
z	-40	-60	0	0	0	0	

Row operations: $L_2 \leftarrow L_2/3$; $L_1 \leftarrow L_1 - L_2$; L_3 unchanged; $z \leftarrow z + 40L_2$ (with the new L_2).

Pivot 2. The only negative number is $-\frac{100}{3}$: x_2 enters. Ratios $20/\frac{4}{3} = 15$, $20/\frac{2}{3} = 30$, $18/1 = 18$: s_1 leaves.

Tableau 1	x_1	x_2	s_1	s_2	s_3	RHS	ratio
s_1	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	0	20	15
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	20	30
s_3	0	1	0	0	1	18	18
z	0	$-\frac{100}{3}$	0	$\frac{40}{3}$	0	800	

Row operations: $L_1 \leftarrow \frac{3}{4}L_1$; $L_2 \leftarrow L_2 - \frac{2}{3}L_1$; $L_3 \leftarrow L_3 - L_1$; $z \leftarrow z + \frac{100}{3}L_1$ (with the new L_1).

Tableau 2	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	15
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	10
s_3	0	0	$-\frac{3}{4}$	$\frac{1}{4}$	1	3
z	0	0	25	5	0	1300

Answer and comparison. No negative number: optimal, $x_1 = 10$, $x_2 = 15$, $s_3 = 3$, $z = 1300$: corner B, as in question 1. The path is $O \rightarrow A \rightarrow B$: **2 pivots**. The lecture (with x_2 first) followed $O \rightarrow D \rightarrow C \rightarrow B$: 3 pivots. Both paths reach the same corner with the same last z -line (25 and 5); only the order of the lines is different. The rule “largest coefficient” is simple, but it is not always the fastest.

