



TD 1 — Solutions

Lecture 1 · Step-by-step solutions



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Operations Research (O.R.)

Every solution follows the method of the lecture: data, variables, objective, constraints, checks. Optimal values are not given for Exercises 2, 3 and 5: you will find them in TD 2 and in later tutorials. Rules refer to the written lecture of Lecture 1.

Solution 1 Linear or not?

Item	Allowed in an LP as written?	Explanation
1. $3x_1 - 2x_2 \leq 7$	Yes	Numbers times variables, added, compared with a number.
2. $10/x_1 + 20/x_2 \leq 4$	No	A division by a variable is not linear.
3. $x_1x_2 \leq 50$	No	A product of two variables. It cannot be fixed: (50, 1) and (1, 50) satisfy it, but their middle point (25.5, 25.5) gives $650.25 > 50$, while linear constraints always keep middle points (Lecture 1, Section 10).
4. $x_1/(x_1 + x_2 + x_3) \leq 0.1$	No (fixable)	The total is positive: multiply by it (Rule 1.1): $x_1 \leq 0.1x_1 + 0.1x_2 + 0.1x_3$, i.e. $0.9x_1 - 0.1x_2 - 0.1x_3 \leq 0$.
5. $ x_1 - x_2 \leq 3$	No (fixable)	Rule 1.7: $x_1 - x_2 \leq 3$ and $x_2 - x_1 \leq 3$. Warning: $ x_1 - x_2 \geq 3$ cannot be fixed.
6. $x_1 < 5$	No	It is a linear inequality, but strict inequalities are not allowed in an LP (a best plan may not exist: 4.9, 4.99, ...). Use $x_1 \leq 5$, or follow Method 1.2 if the value 5 is really forbidden: $x_1 \leq 4$ for whole numbers, $x_1 \leq 5 - \varepsilon$ with a small margin for real numbers.
7. $\min 3x_1 + 2x_2 + 10$	Yes	The expression is affine (linear plus a constant), which is allowed. The constant 10 does not change the best plan: minimise $3x_1 + 2x_2$ and add 10 to the optimal value.
8. $\max \min(x_1, x_2)$	No (fixable)	Rule 1.8: maximise t subject to $t \leq x_1$ and $t \leq x_2$; at the best plan, $t = \min(x_1, x_2)$.
9. $x^3 \leq 8, x \geq 0$	No (fixable)	For $x \geq 0$, x^3 grows when x grows, and $2^3 = 8$: so $x^3 \leq 8$ is the same as $x \leq 2$.

Solution 2 Control boards for robots

1. Data table.

Resource	Basic (per board)	Pro (per board)	Available per day
Assembly machine (min)	2	4	480
Test bench (min)	3	2	360
Special sensors (units)	0	1	100
Profit (thousand DA)	12	20	—

2. Model. Let x_1 = number of Basic boards produced per day and x_2 = number of Pro boards produced per day.

$$\begin{aligned} \max \quad & z = 12x_1 + 20x_2 \quad (\text{daily profit, thousand DA}) \\ \text{s.t.} \quad & 2x_1 + 4x_2 \leq 480 \quad (\text{assembly machine, min}) \\ & 3x_1 + 2x_2 \leq 360 \quad (\text{test bench, min}) \\ & x_2 \leq 100 \quad (\text{sensors}) \\ & x_1, x_2 \geq 0 \end{aligned}$$

Units in the first constraint: (minutes per board) $\times$ (boards per day) = minutes per day, compared with 480 minutes per day.

3. Matrix form. $c = (12, 20)^\top$, $b = (480, 360, 100)^\top$ and

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 2 \\ 0 & 1 \end{pmatrix}.$$

4. Testing the plans.

Plan (x_1, x_2)	Machine (≤ 480)	Bench (≤ 360)	Sensors (≤ 100)	Feasible?	Profit
(50, 100)	500	350	100	No (machine)	—
(80, 60)	400	360	60	Yes	2160
(100, 30)	320	360	30	Yes	1800

(Profits in thousand DA.) For example, for (50, 100): $2 \cdot 50 + 4 \cdot 100 = 500 > 480$; for (80, 60): profit = $12 \cdot 80 + 20 \cdot 60 = 2160$ thousand DA. One violated constraint is enough to make a plan infeasible.

5. Divisibility. The linear program may give fractional numbers of boards. Here the numbers are large (tens of boards), so rounding a fractional solution down keeps it feasible and changes the profit by at most $12 + 20 = 32$ thousand DA. The approximation is acceptable.

Solution 3 Charging a battery

1. Variables and model. Let t_F = number of hours on the fast charger and t_S = number of hours on the slow charger.

$$\begin{aligned} \min \quad & z = 300t_F + 80t_S \quad (\text{cost, DA}) \\ \text{s.t.} \quad & 2t_F + t_S \geq 24 \quad (\text{energy, kWh}) \\ & t_F + t_S \leq 16 \quad (\text{time, hours; one charger at a time}) \\ & t_F, t_S \geq 0 \end{aligned}$$

Units: (kWh per hour) $\times$ (hours) = kWh. The slow charger is cheaper per kWh (80 DA against 150 DA), but alone it is too slow: 24 kWh would need 24 hours $>$ 16 hours.

2. Testing the plans.

Plan (t_F, t_S)	Energy (≥ 24)	Time (≤ 16)	Feasible?	Cost (DA)
(12, 0)	24	12	Yes	3600
(10, 4)	24	14	Yes	$3000 + 320 = 3320$
(6, 10)	22	16	No (energy)	—

Using only the fast charger is feasible but expensive; replacing some fast hours by slow hours already reduces the cost.

3. Form $\min c^T x, Ax \geq b, x \geq 0$. The time constraint is multiplied by -1 (Rule 1.3): $-t_F - t_S \geq -16$. With $x = (t_F, t_S)^T$:

$$c = \begin{pmatrix} 300 \\ 80 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 24 \\ -16 \end{pmatrix}.$$

4. Equalities. Add a surplus variable $e \geq 0$ to the energy constraint and a slack variable $s \geq 0$ to the time constraint (Rule 1.6):

$$2t_F + t_S - e = 24, \quad t_F + t_S + s = 16, \quad t_F, t_S, e, s \geq 0.$$

e is the energy given above the 24 kWh required; s is the number of hours when no charger is used.

Solution 4 Shifts in a control centre

1. Coverage. An operator works during the period in which they start and during the next one. The period 08–12 (period 3) is covered by x_2 (start at 04:00) and x_3 (start at 08:00).

Period	Covered by	Needed
1 (00–04)	x_6, x_1	3
2 (04–08)	x_1, x_2	5
3 (08–12)	x_2, x_3	9
4 (12–16)	x_3, x_4	8
5 (16–20)	x_4, x_5	6
6 (20–24)	x_5, x_6	4

2. Model. One constraint per period ($\geq$, because more operators than needed is allowed):

$$\begin{aligned} \min \quad & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 && \text{(operators per day)} \\ \text{s.t.} \quad & x_6 + x_1 \geq 3, & x_1 + x_2 \geq 5, & x_2 + x_3 \geq 9, \\ & x_3 + x_4 \geq 8, & x_4 + x_5 \geq 6, & x_5 + x_6 \geq 4, \\ & x_j \geq 0, & j = 1, \dots, 6. \end{aligned}$$

The first constraint uses x_6 : an operator who starts at 20:00 still works from 00:00 to 04:00.

3. Hypothesis. Operators are people, so the variables must be integers: divisibility is not satisfied, and the numbers are small. The linear program is still useful: its optimal value is a lower bound for the true minimum, and here (part 4) it even gives an integer schedule.

4. Bonus. Add the constraints of periods 1, 3 and 5:

$$(x_6 + x_1) + (x_2 + x_3) + (x_4 + x_5) \geq 3 + 9 + 6 = 18.$$

The left side is the total number of operators, so $z \geq 18$ for every feasible solution. The schedule $x = (3, 3, 6, 2, 4, 0)$ gives 3, 6, 9, 8, 6 and 4 operators in the six periods, so it is feasible and uses exactly 18 operators: it is optimal. (Other optimal schedules exist.)

Solution 5 Transporting batteries

1. Model. Let x_{ij} = number of units sent per week from factory F_i to city C_j ($i = 1, 2; j = 1, 2, 3$).

$$\begin{aligned} \min \quad & z = 4000x_{11} + 6000x_{12} + 9000x_{13} + 7000x_{21} + 5000x_{22} + 3000x_{23} && \text{(DA per week)} \\ \text{s.t.} \quad & x_{11} + x_{12} + x_{13} \leq 50, & x_{21} + x_{22} + x_{23} \leq 60 && \text{(capacities)} \\ & x_{11} + x_{21} \geq 30, & x_{12} + x_{22} \geq 40, & x_{13} + x_{23} \geq 35 && \text{(demands)} \\ & x_{13} \leq 10 && \text{(road works F1 to C3)} \\ & x_{ij} \geq 0 \end{aligned}$$

2. Summation form. With capacities a_i , demands d_j , unit costs c_{ij} and route limits k_{ij} (here only $k_{13} = 10$ is finite):

$$\min \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad \text{s.t.} \quad \sum_{j=1}^n x_{ij} \leq a_i \quad (\forall i), \quad \sum_{i=1}^m x_{ij} \geq d_j \quad (\forall j), \quad 0 \leq x_{ij} \leq k_{ij}.$$

3. Size. 6 variables; 2 capacity + 3 demand + 1 road constraint = 6 constraints.

4. Special order. The total demand becomes $30 + 40 + 70 = 140$ and the total capacity is $50 + 60 = 110$. If x were feasible, summing the demand constraints and then the capacity constraints would give

$$140 = \sum_j d_j \leq \sum_j \sum_i x_{ij} = \sum_i \sum_j x_{ij} \leq \sum_i a_i = 110,$$

a contradiction. So the problem is infeasible. To still get a useful plan, add variables $u_j \geq 0$ = demand of city j that is not satisfied, with a penalty p_j per unit (larger than any transport cost):

$$\min \sum_i \sum_j c_{ij} x_{ij} + \sum_j p_j u_j \quad \text{s.t.} \quad \sum_i x_{ij} + u_j \geq d_j \quad (\forall j), \quad u_j \geq 0, \quad \text{other constraints unchanged.}$$

This model is always feasible (take all $x_{ij} = 0$ and $u_j = d_j$), and its solution says which cities are not fully served.

Solution 6 A charging depot with a storage battery

1. Variables ($t = 1, 2, 3$, in kWh): g_t = energy from the grid in slot t ; e_t = energy from the generator in slot t ; s_t = energy in the storage battery at the end of slot t ; and $s_0 = 0$.

2. Energy balance (stored at the start + received – given to the robots = stored at the end):

$$s_{t-1} + g_t + e_t - d_t = s_t, \quad t = 1, 2, 3,$$

that is $g_1 + e_1 - s_1 = 120$, $s_1 + g_2 + e_2 - s_2 = 160$, $s_2 + g_3 + e_3 - s_3 = 200$. The constraint $s_t \geq 0$ means that we never use energy that we do not have: the demand of a slot cannot be paid with energy of a later slot.

3. Model.

$$\begin{aligned} \min \quad & z = 5(g_1 + g_2 + g_3) + 6.5(e_1 + e_2 + e_3) + 0.4(s_1 + s_2 + s_3) & (\text{DA}) \\ \text{s.t.} \quad & g_1 + e_1 - s_1 = 120, \quad s_1 + g_2 + e_2 - s_2 = 160, \quad s_2 + g_3 + e_3 - s_3 = 200, \\ & g_t \leq 150, \quad e_t \leq 40, \quad s_t \leq 100, \quad g_t, e_t, s_t \geq 0 \quad (t = 1, 2, 3). \end{aligned}$$

4. Why store early? In slot 3 we can receive at most $150 + 40 = 190$ kWh, but the robots need 200 kWh. So at least 10 kWh must be stored during the earlier slots.

5. A feasible plan. Take $g = (150, 150, 150)$ and $e = (0, 0, 30)$. The balances give $s_1 = 150 - 120 = 30$, $s_2 = 30 + 150 - 160 = 20$ and $s_3 = 20 + 150 + 30 - 200 = 0$, all between 0 and 100. The cost is

$$z = 5 \cdot 450 + 6.5 \cdot 30 + 0.4 \cdot (30 + 20 + 0) = 2250 + 195 + 20 = 2465 \text{ DA.}$$

This plan is in fact the cheapest; the simplex method and duality will let us prove it later in the course. Intuition: grid energy kept for one slot costs $5 + 0.4 = 5.4$ DA/kWh, for two slots 5.8 DA/kWh, both less than the generator (6.5 DA/kWh); so we use all the grid energy first, and the generator only for what is missing in slot 3.

Solution 7 Changing the form of a linear program

1. Standard form of example (b). Rule 1.2: maximise $-3x_1 - 2x_2$. Rule 1.5: $x_2 = x_2^+ - x_2^-$. Rule 1.6: subtract a surplus $e_1 \geq 0$ from the “ $\geq$ ” constraint and add a slack $s_2 \geq 0$ to the “ $\leq$ ” constraint:

$$\begin{aligned} \max \quad & -3x_1 - 2x_2^+ + 2x_2^- \\ \text{s.t.} \quad & x_1 + x_2^+ - x_2^- - e_1 = 2, \\ & x_1 - 2x_2^+ + 2x_2^- + s_2 = 4, \\ & x_1, x_2^+, x_2^-, e_1, s_2 \geq 0. \end{aligned}$$

(The best plan of the original problem is $(x_1, x_2) = (0, 2)$ with $z = 4$; in the standard form it is $x_1 = 0, x_2^+ = 2, x_2^- = 0, e_1 = 0, s_2 = 8$.)

2. Canonical form. Rule 1.2: max $-2x_1 - 3x_2 - x_3$. Rule 1.5: $x_3 = x_3^+ - x_3^-$. Rule 1.3 on the first constraint, Rule 1.4 on the equality (two inequalities, the second multiplied by -1):

$$\begin{aligned} \max \quad & -2x_1 - 3x_2 - x_3^+ + x_3^- \\ \text{s.t.} \quad & -x_1 - x_2 - x_3^+ + x_3^- \leq -4, \\ & 2x_1 - x_2 \leq 3, \quad -2x_1 + x_2 \leq -3, \\ & x_1 - x_3^+ + x_3^- \leq 5, \\ & x_1, x_2, x_3^+, x_3^- \geq 0. \end{aligned}$$

Standard form. Start again from the original problem: the equality stays an equality; subtract a surplus from the “ $\geq$ ” constraint and add a slack to the “ $\leq$ ” constraint:

$$\begin{aligned} \max \quad & -2x_1 - 3x_2 - x_3^+ + x_3^- \\ \text{s.t.} \quad & x_1 + x_2 + x_3^+ - x_3^- - e_1 = 4, \\ & 2x_1 - x_2 = 3, \\ & x_1 - x_3^+ + x_3^- + s_3 = 5, \\ & x_1, x_2, x_3^+, x_3^-, e_1, s_3 \geq 0. \end{aligned}$$

(This problem has a best plan: $(x_1, x_2, x_3) = (1.5, 0, 2.5)$ with $z = 5.5$. Changing the form does not change the best plan; it only changes how the problem is written.)

3. Control boards in standard form.

$$\begin{aligned} \max \quad & 12x_1 + 20x_2 \\ \text{s.t.} \quad & 2x_1 + 4x_2 + s_1 = 480, \quad 3x_1 + 2x_2 + s_2 = 360, \quad x_2 + s_3 = 100, \quad x_1, x_2, s_1, s_2, s_3 \geq 0. \end{aligned}$$

For the plan $(80, 60)$: $s_1 = 480 - 160 - 240 = 80$ minutes of the assembly machine are not used; $s_2 = 360 - 240 - 120 = 0$: the test bench is used completely; $s_3 = 100 - 60 = 40$ special sensors are not used.