



TD 1 — Formulating Linear Programs

Lecture 1 · Exercise sheet



Dr. Yahia Djemmada

National Higher School of Mathematics (NHSM), Algiers

yahia.djemmada@nhsm.edu.dz

Operations Research (O.R.)

Goal of this session: write linear programs from problems described in words, and use the results of Lecture 1. For every model, (1) define the decision variables in full sentences with their units, (2) write the objective, (3) write the constraints, then (4) check signs, units and linearity. Do not solve the models yet: Exercises 2 and 3 will be solved graphically in TD 2, and the others come back when we study the simplex method.

Suggested timing: Exercise 1, 10 min · Exercises 2–5, about 15 min each · Exercises 6 and 7 at home.

Exercise 1 Linear or not? (warm-up)

All letters are real decision variables. For each item, say whether it can appear in a linear program as written. If it cannot, say whether it can be replaced by one or more linear constraints (or a linear objective), and how. Use the rules of Lecture 1.

1. $3x_1 - 2x_2 \leq 7$
2. $10/x_1 + 20/x_2 \leq 4$, with $x_1, x_2 > 0$
3. $x_1x_2 \leq 50$, with $x_1, x_2 \geq 0$
4. $x_1/(x_1 + x_2 + x_3) \leq 0.1$, knowing that $x_1 + x_2 + x_3 > 0$
5. $|x_1 - x_2| \leq 3$
6. $x_1 < 5$
7. Minimise $z = 3x_1 + 2x_2 + 10$
8. Maximise $z = \min(x_1, x_2)$
9. $x^3 \leq 8$, where $x \geq 0$ is a single variable

Exercise 2 Control boards for robots (production)

A workshop produces two electronic control boards for robots, Basic and Pro.

- ▶ Each Basic board needs 2 minutes on the assembly machine and 3 minutes on the test bench.
- ▶ Each Pro board needs 4 minutes on the assembly machine and 2 minutes on the test bench.
- ▶ Each day, the assembly machine is available for 480 minutes and the test bench for 360 minutes.
- ▶ Each Pro board needs one special sensor, and 100 sensors are delivered per day.
- ▶ The profit is 12 thousand DA per Basic board and 20 thousand DA per Pro board. Everything produced is sold.

Questions:

1. Summarise the data in a table (resources in rows, products in columns, with units).
2. Formulate the linear program that maximises the daily profit.
3. Write it in matrix form: maximise $c^T x$ subject to $Ax \leq b$, $x \geq 0$. Give c , A and b .
4. Are the plans $(x_1, x_2) = (50, 100)$, $(80, 60)$ and $(100, 30)$ feasible? Give the profit of each feasible plan.
5. Boards are whole objects. Which hypothesis of linear programming is not satisfied? Does it matter much here?

Exercise 3 Charging a battery (minimum cost)

A battery must receive at least 24 kWh of energy within the next 16 hours. Two chargers can be used, but only one at a time:

Charger	Energy given (kWh per hour)	Cost (DA per hour)
Fast	2	300
Slow	1	80

We want to choose how many hours to use each charger, at minimum cost.

Questions:

1. Define the decision variables (with units) and formulate the linear program.
2. Are the plans $(t_F, t_S) = (12, 0)$, $(10, 4)$ and $(6, 10)$ feasible, where t_F and t_S are the hours on the fast and on the slow charger? Give the cost of each feasible plan.
3. Write the problem in the form: minimise $c^T x$ subject to $Ax \geq b$, $x \geq 0$ (use Rule 1.3). Give c , A and b .
4. Write all the constraints as equalities with non-negative variables, using one surplus variable and one slack variable (Rule 1.6). What does each new variable mean?

Exercise 4 Shifts in a control centre (scheduling)

A control centre monitors a fleet of delivery robots 24 hours a day. The day is split into six periods of 4 hours, each with a minimum number of operators:

Period	00–04	04–08	08–12	12–16	16–20	20–24
Operators needed	3	5	9	8	6	4

Each operator works 8 consecutive hours, starting at 00:00, 04:00, 08:00, 12:00, 16:00 or 20:00. Shifts continue after midnight: an operator who starts at 20:00 works 20–24 and 00–04. We want to employ as few operators per day as possible.

Questions:

1. Let x_j be the number of operators who start at the beginning of period j ($j = 1$ for 00:00, ..., $j = 6$ for 20:00). Which variables cover the period 08–12?
2. Formulate the linear program.
3. Which hypothesis of linear programming is not satisfied? Why can the linear program still be useful?

4. *Bonus*: by adding three of your constraints, prove that at least 18 operators are needed.

Exercise 5 Transporting batteries (transport)

Two factories F1 and F2 produce batteries for robots, and three cities C1, C2, C3 order them. The table gives the weekly capacities, the weekly demands and the transport cost of one unit.

Cost (DA per unit)	City C1	City C2	City C3	Capacity (units per week)
Factory F1	4000	6000	9000	50
Factory F2	7000	5000	3000	60
Demand (units per week)	30	40	35	—

Because of road works, at most 10 units per week can be sent from F1 to C3.

Questions:

1. Formulate the linear program that minimises the weekly transport cost, writing every term explicitly.
2. Write the model in summation notation for m factories and n cities.
3. Count the variables and the constraints (sign constraints not counted).
4. A special order raises the demand of C3 to 70 units. Explain why the problem has no feasible solution (compare the total demand with the total capacity). Propose a modification of the model that still gives a useful answer. *Hint: allow some demand to stay unsatisfied, at a penalty.*

Exercise 6 A charging depot with a storage battery (planning over time, at home)

A depot charges a fleet of delivery robots. The day is split into three slots of 8 hours.

Slot	1	2	3
Charging demand of the robots (kWh)	120	160	200

- ▶ Energy from the grid: at most 150 kWh per slot, at 5 DA/kWh.
- ▶ Energy from a generator: at most 40 kWh per slot, at 6.5 DA/kWh.
- ▶ A storage battery can keep energy for later slots, up to 100 kWh. Keeping energy costs 0.4 DA per kWh stored at the end of a slot.
- ▶ The storage battery is empty at the start, and the demand of each slot must be met in that slot. Ignore energy losses.

Questions:

1. Define, for each slot t , the variables g_t (energy from the grid), e_t (energy from the generator) and s_t (energy in the storage battery at the end of the slot), in kWh.
2. Write the energy balance of each slot, linking s_{t-1} , g_t , e_t , the demand and s_t . What does the constraint $s_t \geq 0$ mean?
3. Formulate the complete linear program.
4. Without solving, explain why some energy must be stored before the slot in which it is used.
5. Give one feasible plan and compute its cost.

Exercise 7 Changing the form of a linear program (at home)

Use Rules 1.2 to 1.6 of Lecture 1.

1. Write in *standard form* the example (b) of the lecture:

$$\min 3x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2 \geq 2, \quad x_1 - 2x_2 \leq 4, \quad x_1 \geq 0, \quad x_2 \text{ free.}$$

2. Write in *canonical form*, then in *standard form*:

$$\begin{aligned} \min z &= 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad &x_1 + x_2 + x_3 \geq 4, \quad 2x_1 - x_2 = 3, \quad x_1 - x_3 \leq 5, \quad x_1, x_2 \geq 0, \quad x_3 \text{ free.} \end{aligned}$$

3. Write the model of Exercise 2 (control boards) in standard form, with slack variables s_1, s_2, s_3 . Give the values of the slack variables for the plan $(80, 60)$, and explain what they mean.

Checklist before you hand in a model

Variables defined with units · units consistent in every constraint · ratios multiplied out (positive denominator) · correct direction of every inequality · no forgotten constraint (signs, balances, limits) · no strict inequality, product or power of variables.