

Lecture 4 — Starting the Simplex Method

The two-phase method, infeasible problems and degeneracy



Dr. Yahia Djemmada

National Higher School of Mathematics (NHSM), Algiers

yahia.djemmada@nhsm.edu.dz

Operations Research (O.R.)

What you will learn. In Lecture 3 the simplex method started at the origin, so the origin had to be a feasible corner. Today we remove this condition. At the end of this lecture you can:

- ▶ prepare any linear program for the simplex method, with slack, surplus and artificial variables;
- ▶ find a first corner with Phase 1, then solve the problem with Phase 2 (the two-phase method);
- ▶ recognise from Phase 1 that a problem has no feasible plan;
- ▶ recognise a degenerate corner and a degenerate pivot, and know the rule that prevents the method from turning in a circle.

With today's lecture, the simplex method can solve every linear program and recognise every case of Lecture 2.

How this lecture is organised. The lecture has six parts, the same six parts as the slides, with the same titles. Each part starts with a banner that shows this map, so you always know where you are:



We use two examples of the earlier lectures: the spare-part boxes (Lecture 2) and the drone workshop. Every tableau and every row operation is written in full. Definitions, rules, methods and key facts are in boxes, each with the reason why it is true. Each part ends with a short summary (“in short”) and a “Check yourself” question; the answers are at the end.

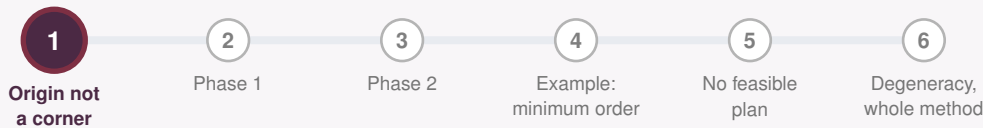
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Part 1 of 6

The origin is not always a corner



In this part, you learn to:

- ▶ see why the origin can be outside the feasible set;
- ▶ prepare any problem with slack, surplus and artificial variables (Rule 4.1);
- ▶ find the first basic variables.

1 Where we are: the start of Lecture 3

In Lecture 3 we solved problems of the form $\max c^T x, Ax \leq b, x \geq 0$, with $b \geq 0$. After adding one slack variable per constraint, the slacks gave us the first corner for free: $x = 0$ and $s = b$, which is ≥ 0 . All the constraints were “resources we must not exceed”.

But many problems contain other kinds of constraints:

- ▶ “ $\geq$ ” constraints: a need, a minimum production, an order to deliver;
- ▶ “ $=$ ” constraints: use exactly all the material, deliver exactly the demand;
- ▶ constraints with a negative right-hand side.

Then the origin is usually *not feasible*, and the simplex method has no corner to start from. The idea of today is surprising and simple: *we use the simplex method itself to find a first corner*.

2 The problem: the origin is not a corner

Recall the spare-part boxes of Lecture 2: a box A contains 2 motors and 1 sensor and costs 30 thousand DA; a box B contains 1 motor and 2 sensors and costs 40 thousand DA; we need at least 8 motors and 7 sensors.

$$\min z = 30x_1 + 40x_2 \quad \text{s.t.} \quad 2x_1 + x_2 \geq 8 \text{ (motors)}, \quad x_1 + 2x_2 \geq 7 \text{ (sensors)}, \quad x_1, x_2 \geq 0.$$

In Lecture 2 we found the answer by drawing: 3 boxes A and 2 boxes B, for 170 thousand DA. Let us try the simplex method. For a “ $\geq$ ” constraint, Rule 1.6 of Lecture 1 says: subtract a surplus variable. We get

$$2x_1 + x_2 - e_1 = 8, \quad x_1 + 2x_2 - e_2 = 7, \quad x_1, x_2, e_1, e_2 \geq 0.$$

If we start at the origin as in Lecture 3 ($x_1 = x_2 = 0$), the equations give $e_1 = -8$ and $e_2 = -7$. These values are negative: the origin is not feasible. On the picture (Figure 1), the origin is outside the shaded region: buying nothing does not give the motors and sensors we need. So the surplus variables cannot be the first basic variables, and we need another way to start.

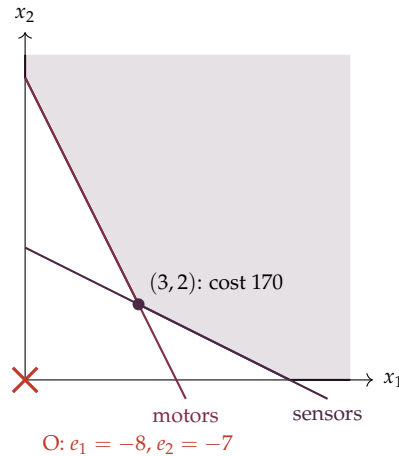


Figure 1: Spare-part boxes: the feasible set (shaded, it goes on without limit) does not contain the origin O. The best plan is the corner (3, 2).

3 Rule 4.1: prepare the equations

The simplex method needs, in every line of the tableau, one variable with coefficient 1 that appears in no other line and whose value (the right-hand side) is ≥ 0 . These variables form the first basis. Rule 4.1 builds such a basis for any linear program.

Rule 4.1 — Prepare the equations

1. If a right-hand side is negative, multiply the whole constraint by -1 (a " $\leq$ " becomes a " $\geq$ ", and conversely).
2. Then, line by line:

Constraint (with $b \geq 0$)	we write	basic at the start
$\dots \leq b$	$\dots + s = b$ (slack)	$s = b$
$\dots \geq b$	$\dots - e + a = b$ (surplus, artificial)	$a = b$
$\dots = b$	$\dots + a = b$ (artificial)	$a = b$

Why does it work? After step 1 every right-hand side is ≥ 0 . In step 2 each line receives one new variable with coefficient $+1$ (a slack or an artificial variable), which appears in no other line. Setting all the other variables to 0, these new variables are equal to the right-hand sides, so they are ≥ 0 : we have a first basic feasible solution, but of a *new* system, the one with the artificial variables.

Why not the surplus? The surplus of a " $\geq$ " line has the coefficient -1 : as a basic variable it would be equal to $-b \leq 0$. This is exactly what went wrong with the spare-part boxes.

For the spare-part boxes, both lines are " $\geq$ " with positive right-hand sides:

$$2x_1 + x_2 - e_1 + a_1 = 8, \quad x_1 + 2x_2 - e_2 + a_2 = 7, \quad \text{all variables} \geq 0.$$

At the start, $x_1 = x_2 = e_1 = e_2 = 0$, so $a_1 = 8$ and $a_2 = 7$.

A second example. For $\min x_1 + x_2$ s.t. $x_1 + 2x_2 \geq 4$, $x_1 - x_2 = 1$, $x_1 \leq 5$, $x \geq 0$, Rule 4.1 gives

$$x_1 + 2x_2 - e_1 + a_1 = 4 \text{ ("} \geq \text{"}, \quad x_1 - x_2 + a_2 = 1 \text{ ("} = \text{"}, \quad x_1 + s_3 = 5 \text{ ("} \leq \text{"},$$

and the first basic variables are $a_1 = 4$, $a_2 = 1$, $s_3 = 5$. The " $\leq$ " line needs no artificial variable: its slack is enough.

Part 1 in short

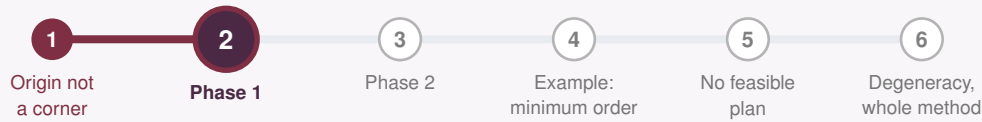
With “ $\geq$ ” or “ $=$ ” constraints, or a negative right-hand side, the origin can be outside the feasible set, and the slacks cannot start. Rule 4.1: first make every $b \geq 0$; then a slack for “ $\leq$ ”, a surplus and an artificial variable for “ $\geq$ ”, an artificial variable for “ $=$ ”. The slacks and artificial variables, equal to b , are the first basis.

Check yourself

Q1. For the second example above, write the Phase 1 objective W (Part 2) and its cleaned line. (Columns $x_1, x_2, e_1, s_3, a_1, a_2$.)

Part 2 of 6

Phase 1: find a first corner



In this part, you learn to:

- ▶ read an artificial variable as a missing quantity;
- ▶ write the Phase 1 objective W and clean its line (Rule 4.2);
- ▶ run Phase 1 and read its end.

4 Definition 4.1: artificial variables

Definition 4.1 — Artificial variable

An *artificial variable* $a_i \geq 0$ is added to a “ $\geq$ ” line or to an “ $=$ ” line only to have a first basis. It has no meaning in the real problem. A plan of the new system is a *real* plan only when all the artificial variables are equal to 0.

A useful reading. In the line $2x_1 + x_2 - e_1 + a_1 = 8$, the variable a_1 is what we must add to the motors we buy to reach 8: it is the number of motors that are *missing*. In the same way, a_2 is the number of missing sensors. At the start we buy nothing, so we miss 8 motors and 7 sensors (Figure 2). The start is a fake plan, and our job is to push the artificial variables down to 0.

Start: buy nothing

motors $a_1 = 8$ missing

sensors $a_2 = 7$ missing

■ we have □ missing = artificial variable

Figure 2: At the start of Phase 1 nothing is bought: all the motors and sensors are missing.

5 Phase 1: make the missing quantities zero

To push the artificial variables to 0, we solve a first linear program, called *Phase 1*:

$$\max W = -(a_1 + a_2) \quad \text{subject to the prepared equations, all variables} \geq 0.$$

Since $a_1, a_2 \geq 0$, we always have $W \leq 0$, and $W = 0$ exactly when $a_1 = a_2 = 0$. In words: Phase 1 makes the total of the missing quantities as small as possible (Figure 3). For this new problem, the start ($a = b$, all other variables 0) is a corner, so we can use the simplex method of Lecture 3 without any change.

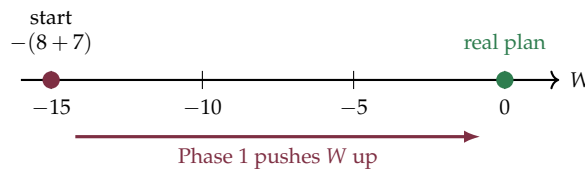


Figure 3: The value of W is never positive. It is -15 at the start, and 0 exactly for a real plan.

Key fact 4.1

The real problem has a feasible plan if and only if the best value of W in Phase 1 is 0.

Why? Suppose the real problem has a feasible plan x . Put all the artificial variables equal to 0 (and the slacks and surpluses equal to their values for x): the equations hold, and $W = 0$. Since W is never positive, 0 is the best possible value. Conversely, if the best W is 0, then all the artificial variables are 0, and the equations become exactly the real constraints: the plan is feasible. Note also that Phase 1 can never be unbounded: W can never be larger than 0.

6 Rule 4.2: clean the W -line before the first pivot

We write the first tableau of Phase 1. The objective $W = -a_1 - a_2$ gives the line $W + a_1 + a_2 = 0$ (top of Figure 4). Something is wrong in this tableau: a_1 and a_2 are basic, but their columns are not unit columns, because they have 1 in the W -line. A tableau is read correctly only when every basic column has 0 in the objective line: then the objective line only contains non-basic variables, as in a dictionary, and its RHS is the value of the objective. Here the RHS says $W = 0$, but in fact $W = -(8 + 7) = -15$.

Rule 4.2 — Clean the objective line

Before the first pivot of a phase, the objective line must have 0 under every basic variable. Subtract from it the right multiple of each line whose basic variable has a non-zero number in the objective line. In Phase 1: subtract from the W -line every line that contains an artificial variable.

For the spare-part boxes, $W \leftarrow W - L_1 - L_2$. The bottom of Figure 4 shows the computation, column by column.

Start	x_1	x_2	e_1	e_2	a_1	a_2	RHS
a_1	2	1	-1	0	1	0	8
a_2	1	2	0	-1	0	1	7
W	0	0	0	0	1	1	0

must be 0!

W	0	0	0	0	1	1	0
$-L_1$	-2	-1	1	0	-1	0	-8
$-L_2$	-1	-2	0	1	0	-1	-7
W_{new}	-3	-3	1	1	0	0	-15

Figure 4: Top: the raw first tableau of Phase 1; the two 1s under the basic variables a_1, a_2 are not allowed. Bottom: the cleaning $W - L_1 - L_2$, column by column.

Now the RHS of the W-line is -15 , the correct value at the start. The numbers make sense too: the -3 under x_1 says that one more box A makes W grow by 3, because it gives 2 of the missing motors and 1 of the missing sensors.

7 Phase 1, pivot by pivot

The cleaned tableau T0 is now an ordinary simplex tableau (Rule 3.2). In the figures, the entering column is shaded, the ratios are on the right, the pivot and the smallest ratio are boxed, and the new tableau is below, with the row operation of each line on its right (L_i means the *new* pivot line).

Pivot 1 (Figure 5). The most negative numbers in the W-line are -3 and -3 : a tie, so we take the smallest index, x_1 . Ratios: $8/2 = 4$ and $7/1 = 7$. The smallest is 4, in L_1 : a_1 leaves. Row operations: $L_1 \leftarrow L_1/2$; $L_2 \leftarrow L_2 - L_1$; $W \leftarrow W + 3L_1$. Tableau T1 is the point $(4, 0)$: 4 boxes A give all 8 motors ($a_1 = 0$) but only 4 sensors, so 3 sensors are still missing ($a_2 = 3$), and $W = -3$.

		enters ↓ (tie: x_1)							
	T0	x_1	x_2	e_1	e_2	a_1	a_2	RHS	ratio
leaves	a_1	2	1	-1	0	1	0	8	$8/2 = 4$
	a_2	1	2	0	-1	0	1	7	$7/1 = 7$
	W	-3	-3	1	1	0	0	-15	
		pivot							
	T1	x_1	x_2	e_1	e_2	a_1	a_2	RHS	
	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	4	$L_1 \div 2$
	a_2	0	$\frac{3}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	3	$L_2 - L_1$
	W	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	-3	$W + 3L_1$

Figure 5: Phase 1, pivot 1: x_1 enters, a_1 leaves. T1 is the point $(4, 0)$, still outside the feasible set.

Pivot 2 (Figure 6). The only negative number is $-\frac{3}{2}$, under x_2 : x_2 enters. Ratios: $4/\frac{1}{2} = 8$ and $3/\frac{3}{2} = 2$. The smallest is 2, in L_2 : a_2 leaves. Row operations: $L_2 \leftarrow L_2/\frac{3}{2}$ (that is $\frac{2}{3}L_2$); $L_1 \leftarrow L_1 - \frac{1}{2}L_2$; $W \leftarrow W + \frac{3}{2}L_2$.

		enters ↓							
T1	x_1	x_2	e_1	e_2	a_1	a_2	RHS	ratio	
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	4	$4/\frac{1}{2} = 8$	
leaves a_2	0	$\frac{3}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	3	$3/\frac{3}{2} = 2$	
W	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	-3		
		pivot ↓							
T2	x_1	x_2	e_1	e_2	a_1	a_2	RHS		
x_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	3	$L_1 - \frac{1}{2}L_2$	
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	2	$L_2 \div \frac{3}{2}$	
W	0	0	0	0	1	1	0	$W + \frac{3}{2}L_2$	

Figure 6: Phase 1, pivot 2: x_2 enters, a_2 leaves. In T2 the W-line has no negative number and $W = 0$.

End of Phase 1. There is no negative number in the W-line of T2, so Phase 1 is finished, and its best value is $W = 0$. By Key fact 4.1 the problem has a feasible plan, and the tableau gives one: $a_1 = a_2 = 0$, $x_1 = 3$, $x_2 = 2$, $e_1 = e_2 = 0$. This is a real corner of the feasible set.

8 The walk of Phase 1: from outside to a corner

Point	a_1 (motors missing)	a_2 (sensors missing)	W
O (0,0)	8	7	-15
(4,0)	0	3	-3
(3,2)	0	0	0

Figure 7 shows the walk of Phase 1: it starts outside the feasible set, at the origin ($W = -15$), goes to (4,0), still outside ($W = -3$), and reaches the corner (3,2) ($W = 0$). Phase 1 walks through points of the new system that are not real plans, reducing the missing quantities, until it reaches a corner of the feasible set.

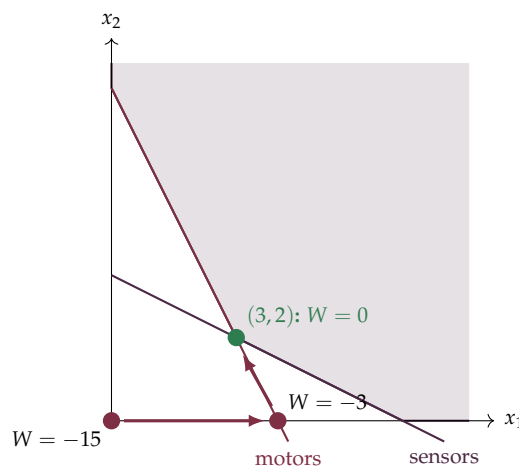


Figure 7: The walk of Phase 1 for the spare-part boxes: $(0,0) \rightarrow (4,0) \rightarrow (3,2)$.

Part 2 in short

An artificial variable measures what is missing in its line; a plan is real only when all are 0 (Definition 4.1). Phase 1 maximises $W = -(\text{sum of the artificial variables})$; the real problem

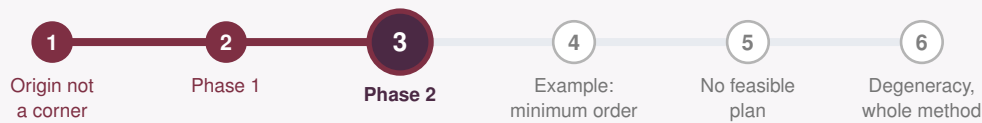
is feasible exactly when the best W is 0 (Key fact 4.1). Before the first pivot, clean the W -line so that it has 0 under the basic variables (Rule 4.2). Then Phase 1 is an ordinary simplex method, which walks from outside to a corner.

Check yourself

Q2. In T0 of Phase 1, the numbers under x_1 and x_2 are both -3 . Choose x_2 as the first entering variable instead. Which variable leaves, which point do you reach, and what is W there? Finish Phase 1.

Part 3 of 6

Phase 2, and the two-phase method



In this part, you learn to:

- ▶ start Phase 2 from the corner found by Phase 1;
- ▶ write and clean the real objective line;
- ▶ read the answer, and the proof in the last line.

9 Phase 2: back to the real objective

Now we have a real corner, and we solve the real problem from there (Figure 8).

1. Delete the artificial columns. The artificial variables are non-basic and equal to 0, and they must stay 0 for the plan to be real. Deleting their columns is the same as fixing them at 0.

2. Write the real objective line. The problem is a minimisation, so we maximise $z' = -z = -30x_1 - 40x_2$ (Lecture 1, Rule 1.2). The line is $z' + 30x_1 + 40x_2 = 0$: the numbers 30 and 40 go in the z' -line.

3. Clean it (Rule 4.2): x_1 and x_2 are basic, so their numbers 30 and 40 must become 0: $z' \leftarrow z' - 30L_1 - 40L_2$. For example, the number under e_1 becomes $0 - 30 \times (-\frac{2}{3}) - 40 \times \frac{1}{3} = 20 - \frac{40}{3} = \frac{20}{3}$, and the RHS becomes $0 - 30 \times 3 - 40 \times 2 = -170$.

4. Read. There is no negative number in the cleaned line: the tableau is already optimal. We read $x_1 = 3$, $x_2 = 2$, $z' = -170$. So the minimum cost is $z = 170$ thousand DA: buy 3 boxes A and 2 boxes B, as in Lecture 2. Both needs are met exactly ($e_1 = e_2 = 0$). Here Phase 2 needed no pivot; in general it needs some (Part 4).

end of Phase 1							
T2	x_1	x_2	e_1	e_2	a_1	a_2	RHS
x_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	3
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	2
W	0	0	0	0	1	1	0

1. delete

Ph. 2	x_1	x_2	e_1	e_2	RHS
x_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	3
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	2
z'	30	40	0	0	0

2. raw line

z'	0	0	$\frac{20}{3}$	$\frac{50}{3}$	-170
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3. clean: $z' - 30L_1 - 40L_2$; 4. optimal

Figure 8: From the end of Phase 1 to the optimal tableau of Phase 2: delete the artificial columns, write the real line, clean it.

10 Reading the answer, and the proof

The cleaned line reads $z' + \frac{20}{3}e_1 + \frac{50}{3}e_2 = -170$, that is $z' = -170 - \frac{20}{3}e_1 - \frac{50}{3}e_2$. So:

1. the cost is $-z' = 170 + \frac{20}{3}e_1 + \frac{50}{3}e_2$;
2. every feasible plan has $e_1 \geq 0$ and $e_2 \geq 0$ (the extra motors and the extra sensors);
3. so every feasible plan costs at least 170, and (3,2) costs exactly 170: it is optimal (Figure 9).

A link with Lecture 2. The numbers $\frac{20}{3}$ and $\frac{50}{3}$ are exactly the multipliers that proved “cost ≥ 170 ” in Lecture 2 (Check yourself Q3 there). As in Lecture 3, the last tableau contains a proof of optimality.

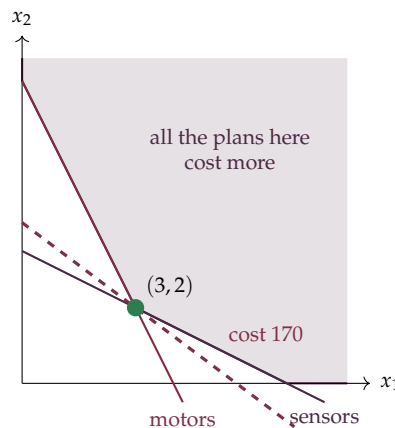


Figure 9: The whole feasible set lies on the expensive side of the cost line $30x_1 + 40x_2 = 170$.

11 Method 4.1: the two-phase method

Method 4.1 — The two-phase method

1. **Prepare** the equations (Rule 4.1): right-hand sides ≥ 0 ; slack, surplus and artificial variables.
2. **Phase 1**: maximise $W = -(\text{sum of the artificial variables})$. Clean the W -line (Rule 4.2) and run the simplex method (Method 3.2).
3. **End of Phase 1**: if $W < 0$, the problem has no feasible plan: stop (Rule 4.3, Part 5). If $W = 0$, the tableau gives a real corner.
4. **Phase 2**: delete the artificial columns, write the line of the real objective (for a minimisation, max $-z$), clean it (Rule 4.2), and run the simplex method.

If no artificial variable is needed (all constraints " $\leq$ " with $b \geq 0$), skip Phase 1: this is Lecture 3.

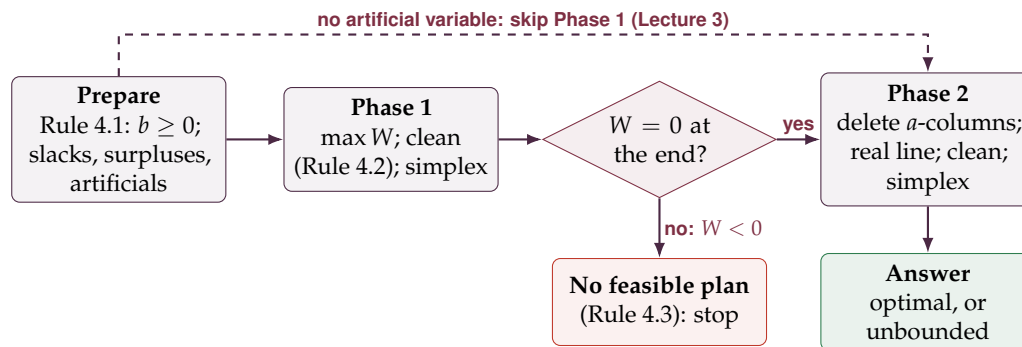


Figure 10: Method 4.1 as a picture.

Part 3 in short

Phase 2 starts from the corner of Phase 1: delete the artificial columns, write the real objective line (for a minimisation, maximise $-z$), clean it with Rule 4.2, and continue with the simplex method. The last line is again a proof: here cost = $170 + \frac{20}{3}e_1 + \frac{50}{3}e_2 \geq 170$, with the multipliers of Lecture 2.

Check yourself

Q3. At the start of Phase 2, what would happen if we forgot to clean the line $z' + 30x_1 + 40x_2 = 0$? What would we (wrongly) conclude?

Part 4 of 6

Example: a minimum order



In this part, you learn to:

- ▶ solve a maximisation with a “ $\geq$ ” constraint;
- ▶ follow Phase 1 and Phase 2 on the picture;
- ▶ read the surplus in the answer.

12 Example 4.2: a minimum order

The problem. The drone workshop receives an order: at least 12 drones B per week. We add the constraint $x_2 \geq 12$:

$$\max z = 40x_1 + 60x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \leq 40, \quad 3x_1 + 2x_2 \leq 60, \quad x_2 \leq 18, \quad x_2 \geq 12, \quad x_1, x_2 \geq 0.$$

The origin is not feasible any more ($0 \geq 12$ is false). The feasible set is the part of the drone workshop above the line $x_2 = 12$ (Figure 12).

Prepare (Rule 4.1): the first three lines get their slacks s_1, s_2, s_3 ; the new line gets a surplus and an artificial variable: $x_2 - e_4 + a_4 = 12$. Only one artificial variable is needed, so Phase 1 is $\max W = -a_4$.

13 Example 4.2: Phase 1 in one pivot

The line $W + a_4 = 0$ is cleaned: $W \leftarrow W - L_4$, which gives -1 under x_2 , $+1$ under e_4 and -12 on the right. Then x_2 enters (the only negative number), the ratios are $40/2 = 20$, $60/2 = 30$, $18/1 = 18$ and $12/1 = 12$, and a_4 leaves (Figure 11). Row operations: $L_1 \leftarrow L_1 - 2L_4$; $L_2 \leftarrow L_2 - 2L_4$; $L_3 \leftarrow L_3 - L_4$; $W \leftarrow W + L_4$.

	enters ↓									
	T0	x_1	x_2	s_1	s_2	s_3	e_4	a_4	RHS	ratio
	s_1	1	2	1	0	0	0	0	40	$40/2 = 20$
	s_2	3	2	0	1	0	0	0	60	$60/2 = 30$
	s_3	0	1	0	0	1	0	0	18	$18/1 = 18$
leaves	a_4	0	1	0	0	0	-1	1	12	$12/1 = 12$
	W	0	-1	0	0	0	1	0	-12	
	↓ pivot									
	T1	x_1	x_2	s_1	s_2	s_3	e_4	a_4	RHS	
	s_1	1	0	1	0	0	2	-2	16	$L_1 - 2L_4$
	s_2	3	0	0	1	0	2	-2	36	$L_2 - 2L_4$
	s_3	0	0	0	0	1	1	-1	6	$L_3 - L_4$
	x_2	0	1	0	0	0	-1	1	12	$L_4 \div 1$
	W	0	0	0	0	0	0	1	0	$W + L_4$

Figure 11: Example 4.2, Phase 1: one pivot. In T1, $W = 0$.

$W = 0$: Phase 1 is finished after one pivot, at the real corner $(0, 12)$: 12 drones B and no drone A, with $s_1 = 16$ assembly hours, $s_2 = 36$ testing hours and $s_3 = 6$ cameras left. The profit there is $z = 60 \times 12 = 720$.

14 Example 4.2: Phase 2, three pivots to B

Delete the column a_4 and write the real line $z - 40x_1 - 60x_2 = 0$. The basic variable x_2 has -60 in it, so we clean it: $z \leftarrow z + 60L_4$.

Tableau 0	x_1	x_2	s_1	s_2	s_3	e_4	RHS	ratio
s_1	1	0	1	0	0	2	16	8
s_2	3	0	0	1	0	2	36	18
s_3	0	0	0	0	1	1	6	6
x_2	0	1	0	0	0	-1	12	—
z	-40	0	0	0	0	-60	720	

Pivot 1. The most negative number is -60 , under e_4 : e_4 enters. In words: make *more* than 12 drones B, since each one adds 60 thousand DA. Ratios: $16/2 = 8$, $36/2 = 18$, $6/1 = 6$; the line x_2 has -1 , so no ratio. The smallest is 6: s_3 leaves (the cameras run out). Row operations: $L_1 \leftarrow L_1 - 2L_3$; $L_2 \leftarrow L_2 - 2L_3$; $L_4 \leftarrow L_4 + L_3$; $z \leftarrow z + 60L_3$.

Tableau 1	x_1	x_2	s_1	s_2	s_3	e_4	RHS	ratio
s_1	1	0	1	0	-2	0	4	4
s_2	3	0	0	1	-2	0	24	8
e_4	0	0	0	0	1	1	6	—
x_2	0	1	0	0	1	0	18	—
z	-40	0	0	0	60	0	1080	

Pivot 2. This is the corner $D = (0, 18)$, with $z = 1080$. Now x_1 enters, the ratios are $4/1 = 4$ and $24/3 = 8$, and s_1 leaves. Row operations: $L_2 \leftarrow L_2 - 3L_1$; $z \leftarrow z + 40L_1$.

Tableau 2	x_1	x_2	s_1	s_2	s_3	e_4	RHS	ratio
x_1	1	0	1	0	-2	0	4	—
s_2	0	0	-3	1	4	0	12	3
e_4	0	0	0	0	1	1	6	6
x_2	0	1	0	0	1	0	18	18
z	0	0	40	0	-20	0	1240	

Pivot 3. This is the corner $C = (4, 18)$, with $z = 1240$. Now s_3 enters (-20); the line x_1 has -2 (no ratio), and the ratios are $12/4 = 3$, $6/1 = 6$ and $18/1 = 18$: s_2 leaves. Row operations: $L_2 \leftarrow L_2/4$; $L_1 \leftarrow L_1 + 2L_2$; $L_3 \leftarrow L_3 - L_2$; $L_4 \leftarrow L_4 - L_2$; $z \leftarrow z + 20L_2$.

Tableau 3	x_1	x_2	s_1	s_2	s_3	e_4	RHS
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	10
s_3	0	0	$-\frac{3}{4}$	$\frac{1}{4}$	1	0	3
e_4	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	0	1	3
x_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	0	15
z	0	0	25	5	0	0	1300

Answer. No negative number: optimal. The plan is $B = (10, 15)$ with $z = 1300$ thousand DA, the same as in Lecture 2. This is normal: the old best plan makes 15 drones B, so it already satisfies the order. The surplus $e_4 = 3$ says that we make 3 drones B more than ordered. Figure 12 shows the whole walk.

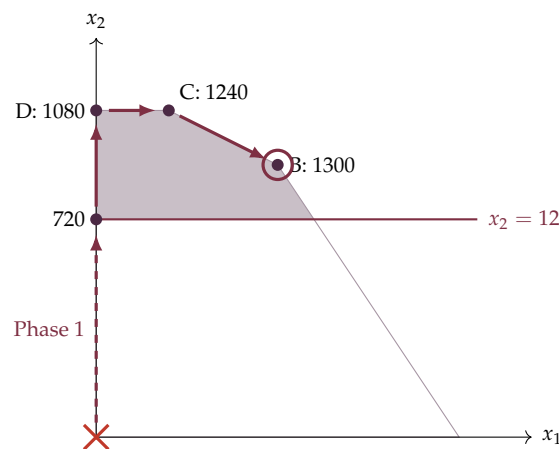


Figure 12: Example 4.2: Phase 1 (dashed) goes from the origin to $(0, 12)$; Phase 2 (solid) goes to D, C and B. The feasible set is the darker part above the line $x_2 = 12$.

Part 4 in short

A “ $\geq$ ” constraint in a maximisation is handled exactly like in a minimisation: one artificial variable, Phase 1 until $W = 0$, then Phase 2 with the real line, cleaned. A surplus can enter in Phase 2: it means “do more than the minimum”. In the answer, the surplus says by how much the order is exceeded.

Check yourself

Q4. The order becomes “at least 16 drones B”. Where does Phase 1 end? Is the old best plan B still feasible? Solve the problem (a drawing is enough to check).

Part 5 of 6

No feasible plan, and special cases



In this part, you learn to:

- ▶ recognise a problem with no feasible plan from Phase 1 (Rule 4.3);
- ▶ treat equalities and negative right-hand sides;
- ▶ handle an artificial variable still basic at the end of Phase 1.

15 Signal 3: Phase 1 ends with $W < 0$

Take the infeasible problem of Lecture 2 (Case 5):

$$\max x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 1, \quad x_1 + x_2 \geq 2, \quad x_1, x_2 \geq 0.$$

Prepare: $x_1 + x_2 + s_1 = 1$ and $x_1 + x_2 - e_2 + a_2 = 2$. **Phase 1:** $\max W = -a_2$; clean the line: $W \leftarrow W - L_2$. Then x_1 enters (tie with x_2 , smallest index), the ratios are $1/1 = 1$ and $2/1 = 2$, and s_1 leaves (Figure 13). Row operations: $L_2 \leftarrow L_2 - L_1$; $W \leftarrow W + L_1$.

		enters ↓						
	T0	x_1	x_2	s_1	e_2	a_2	RHS	ratio
leaves	s_1	1	1	1	0	0	1	1/1 = 1
	a_2	1	1	0	-1	1	2	2/1 = 2
	W	-1	-1	0	1	0	-2	
		pivot ↓						
	T1	x_1	x_2	s_1	e_2	a_2	RHS	
	x_1	1	1	1	0	0	1	$L_1 \div 1$
	a_2	0	0	-1	-1	1	1	$L_2 - L_1$
	W	0	0	1	1	0	-1	$W + L_1$

Figure 13: Phase 1 of an infeasible problem: T1 has no negative number in its W-line, but $W = -1$.

There is no negative number in the W-line of T1: Phase 1 is finished. But the best value is $W = -1$, not 0.

Rule 4.3 — No feasible plan

If Phase 1 ends with $W < 0$, the problem has no feasible plan. Stop: Phase 2 is not needed.

Why? This is Key fact 4.1: the best W is not 0, so there is no real plan.

16 What $W < 0$ tells us

The last tableau has $a_2 = 1$. Since a_2 measures what is missing in the line $x_1 + x_2 \geq 2$, even the best plan of Phase 1, the point $(1, 0)$, misses one unit: $x_1 + x_2 = 1$, and we need 2. On the picture

(Figure 14), the two half-planes do not meet, and the gap between the two lines is this missing unit. In practice, an infeasible problem means an error in the data or in the model, or needs that are too high for the resources. The value of the artificial variable tells you how much is missing.

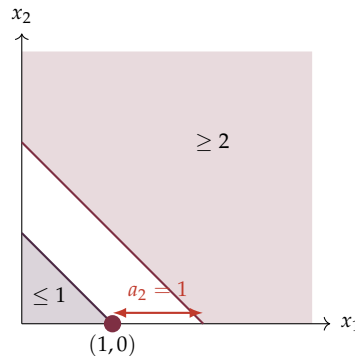


Figure 14: No feasible plan: the half-planes $x_1 + x_2 \leq 1$ and $x_1 + x_2 \geq 2$ do not meet. Phase 1 stops at $(1,0)$, one unit short.

With this signal, the simplex method recognises all the cases of Lecture 2 from its tableaux: no feasible plan ($W < 0$ at the end of Phase 1, Rule 4.3), an unbounded problem (no ratio, Rule 3.3), many optimal plans (a 0 under a non-basic variable in the optimal z -line, Rule 3.4), and a unique optimum otherwise.

17 Equalities and negative right-hand sides

- ▶ **A negative right-hand side.** $x_1 - 2x_2 \leq -2$: multiply by -1 , which gives $-x_1 + 2x_2 \geq 2$, a “ $\geq$ ” line with a positive right-hand side, so $-x_1 + 2x_2 - e + a = 2$.
- ▶ **An equality.** $x_1 + x_2 + x_3 = 6$ becomes $x_1 + x_2 + x_3 + a = 6$: one artificial variable, no slack, no surplus.
- ▶ **Why must b be ≥ 0 first?** Because the starting basic variables are equal to b , and they must be ≥ 0 .

Three things *not* to do:

- ▶ add an artificial variable to a “ $\leq$ ” line with $b \geq 0$: its slack is already a good starting variable;
- ▶ write a surplus with a “ $+$ ” sign;
- ▶ replace an equality by two inequalities here (Rule 1.4 was for the canonical form): one artificial variable is simpler.

18 A special end of Phase 1

Sometimes Phase 1 ends with $W = 0$, but an artificial variable is still *basic*, with the value 0. This can happen after a tie in the ratio test. The plan is real (all the artificial variables are 0), but the artificial variable must leave the basis before Phase 2, because we delete its column.

- ▶ Its line has a right-hand side equal to 0. Choose a non-zero number in this line, under a column that is *not* artificial, and pivot on it. No ratio test is needed: the right-hand side is 0, so the pivot changes no value (and a negative pivot is allowed here).

- If the line has only zeros under the non-artificial columns, the constraint was *redundant*: it is a combination of the other constraints. Delete the line.

Then Phase 2 starts as usual. Remember: only $W < 0$ means “no feasible plan”; $W = 0$ always gives a real plan.

Part 5 in short

Phase 1 ending with $W < 0$ means that the problem has no feasible plan (Rule 4.3); the value of the artificial variables says how much is missing. Negative right-hand sides are first multiplied by -1 ; an equality gets one artificial variable. An artificial variable still basic at 0 at the end of Phase 1 is removed by a pivot with RHS 0, or its line is deleted if it is redundant.

Check yourself

Q5. In Example 4.2, the order becomes “at least 20 drones B”. Run Phase 1. What do you conclude, and what does the last value of a_4 mean?

Part 6 of 6

Degeneracy, and the complete method



In this part, you learn to:

- recognise a degenerate corner and a degenerate pivot;
- know Bland’s rule, which prevents cycling;
- see the complete simplex method and all its signals.

19 Degeneracy: a basic variable equal to 0

Normally, at a basic feasible solution, the non-basic variables are 0 and the basic variables are positive. Sometimes a basic variable is 0 too.

Definition 4.2 — Degenerate solution

A basic feasible solution is *degenerate* when at least one *basic* variable is equal to 0.

Where does it come from? After a *tie* in the ratio test: two lines reach 0 at the same time; one basic variable leaves, and the other stays basic with the value 0. Or when a right-hand side is 0 from the start. On the picture, more constraint lines than necessary pass through the corner: this is the degenerate corner of Lecture 2 (special situation B), where three lines meet in the plane (Figure 17).

Why does it matter? At a degenerate solution, the next smallest ratio can be 0. Then the entering variable enters with the value 0: the pivot changes the basis, but not the point, and z does not grow. Such a pivot is called a *degenerate pivot*.

20 A degenerate pivot, then a useful pivot

$$\max z = 2x_1 + x_2 \quad \text{s.t.} \quad x_1 - x_2 \leq 0, \quad x_1 + x_2 \leq 4, \quad x_1, x_2 \geq 0.$$

The first constraint says “at most as many drones A as drones B”. Its right-hand side is 0, so the first tableau is already degenerate: s_1 is basic with the value 0.

Pivot 1: a degenerate pivot (Figure 15). x_1 enters (−2). Ratios: $0/1 = 0$ and $4/1 = 4$. The smallest ratio is 0: s_1 leaves, and x_1 enters with the value 0. Row operations: $L_2 \leftarrow L_2 - L_1$; $z \leftarrow z + 2L_1$. In T1 we are still at the origin, and z is still 0: the basis changed, but not the point.

		enters ↓					
	T0	x_1	x_2	s_1	s_2	RHS	ratio
leaves	s_1	1	−1	1	0	0	0/1 = 0
	s_2	1	1	0	1	4	4/1 = 4
	z	−2	−1	0	0	0	
		pivot ↓					
	T1	x_1	x_2	s_1	s_2	RHS	
	x_1	1	−1	1	0	0	$L_1 \div 1$
	s_2	0	2	−1	1	4	$L_2 - L_1$
	z	0	−3	2	0	0	$z + 2L_1$

Figure 15: A degenerate pivot: the smallest ratio is 0, so T1 is still the origin, with $z = 0$.

Pivot 2: a useful pivot (Figure 16). x_2 enters (−3); the line x_1 has −1 (no ratio), and the ratio in L_2 is $4/2 = 2$: s_2 leaves. Row operations: $L_2 \leftarrow L_2/2$; $L_1 \leftarrow L_1 + L_2$; $z \leftarrow z + 3L_2$. No negative number in T2: optimal, at (2, 2) with $z = 6$.

		enters ↓					
	T1	x_1	x_2	s_1	s_2	RHS	ratio
	x_1	1	−1	1	0	0	−1: no ratio
leaves	s_2	0	2	−1	1	4	4/2 = 2
	z	0	−3	2	0	0	
		pivot ↓					
	T2	x_1	x_2	s_1	s_2	RHS	
	x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	2	$L_1 + L_2$
	x_2	0	1	− $\frac{1}{2}$	$\frac{1}{2}$	2	$L_2 \div 2$
	z	0	0	$\frac{1}{2}$	$\frac{3}{2}$	6	$z + 3L_2$

Figure 16: The next pivot moves to the optimal corner (2, 2), with $z = 6$.

On the picture (Figure 17). Three lines pass through the origin: $x_1 = 0$, $x_2 = 0$ and $x_1 - x_2 = 0$. In the plane, two lines are enough to fix a point, so the origin has several descriptions by its non-basic variables: $\{x_1, x_2\}$ (T0) and $\{x_2, s_1\}$ (T1). Pivot 1 only changes the description of the origin; pivot 2 then moves along the line $x_1 = x_2$ to the corner (2, 2). A degenerate pivot is not wasted: it chooses a description of the same corner from which the method can move.

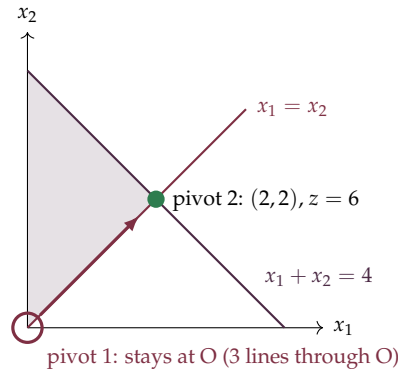


Figure 17: A degenerate corner: three lines pass through the origin. Pivot 1 stays at O; pivot 2 moves to (2,2).

21 Cycling, and Bland's rule

With degenerate pivots, z can stay the same for several pivots in a row. In rare examples, built on purpose (the first one was published by E. Beale in 1955), the method can even come back to a basis that it has already visited. Then it repeats the same pivots forever: this is called *cycling*. Cycling almost never happens in practice, but a simple rule makes it impossible.

Key fact 4.2 — Bland's rule (admitted)

Choose as entering variable the *first* variable (smallest index) with a negative number in the objective line; and as leaving variable, among the lines with the smallest ratio, the one whose basic variable has the smallest index. With this rule, the simplex method always stops after a finite number of pivots.

This rule was found by R. Bland in 1977; we admit it without proof.

In this course we keep our usual rule: the most negative number enters, and ties are broken by the smallest index. If you ever see the same value of z for several pivots in a row, switch to Bland's rule. In Lecture 3 (Part 6, "Why the simplex method works") we said that the method stops because z grows at each pivot; degeneracy was the one exception, and Bland's rule closes it.

22 Other books: the Big-M method

Many books use another method, with one single phase. The artificial variables are put directly in the real objective, with a huge penalty M . For the spare-part boxes:

$$\max -30x_1 - 40x_2 - Ma_1 - Ma_2 \quad (M \text{ is a "very large" number}).$$

Because of the penalty, the simplex method first pushes the artificial variables to 0, as Phase 1 does, and then continues with the real objective. After cleaning, the objective line is $(30 - 3M, 40 - 3M, M, M, 0, 0 \mid -15M)$: to choose the entering variable we must compare expressions like $30 - 3M$ and $40 - 3M$, or choose a numerical value for M , which can cause rounding errors on a computer. The two methods make the same kind of pivots. In this course we use the two-phase method: there are no huge numbers, and the signal " $W < 0$ " for infeasible problems is very clear.

23 The complete simplex method

Lectures 3 and 4 together give a method for every linear program (Figure 18).

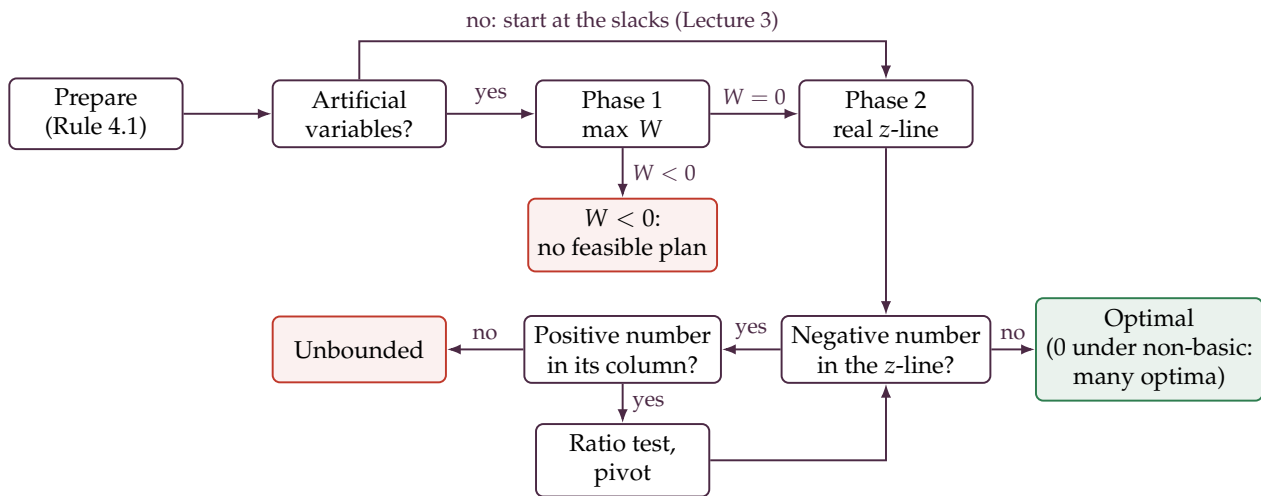


Figure 18: The complete simplex method (Lectures 3 and 4).

All the signals in one table.

Where	What we see	Conclusion
End of Phase 1	$W < 0$	no feasible plan (Rule 4.3)
End of Phase 1	$W = 0$	a real corner: go to Phase 2
Objective line	no negative number	optimal (Key fact 3.2)
Objective line	a 0 under a non-basic variable	many optimal plans (Rule 3.4)
Entering column	no positive number	unbounded (Rule 3.3)
Ratio test	smallest ratio = 0	degenerate pivot: same point

24 Common mistakes

Wrong ✗	Right ✓
Forget to clean the W-line at the start of Phase 1, or the z-line at the start of Phase 2	Clean before the first pivot of each phase (Rule 4.2): 0 under every basic variable, and the RHS is the true value
Add the variables before making the right-hand sides ≥ 0	First multiply by -1 the lines with $b < 0$
Add an artificial variable to a " $\leq$ " line with $b \geq 0$, or write a surplus with a "+" sign	Artificial variables only on " $\geq$ " and " $=$ " lines; a surplus has a "-" sign
Keep the artificial columns in Phase 2	Delete them at the end of Phase 1
Read $z' = -170$ as a cost of -170	For a minimisation: cost = $-z' = 170$
Conclude "no feasible plan" when $W = 0$ with an artificial variable still basic	Only $W < 0$ means no feasible plan (Section 18)

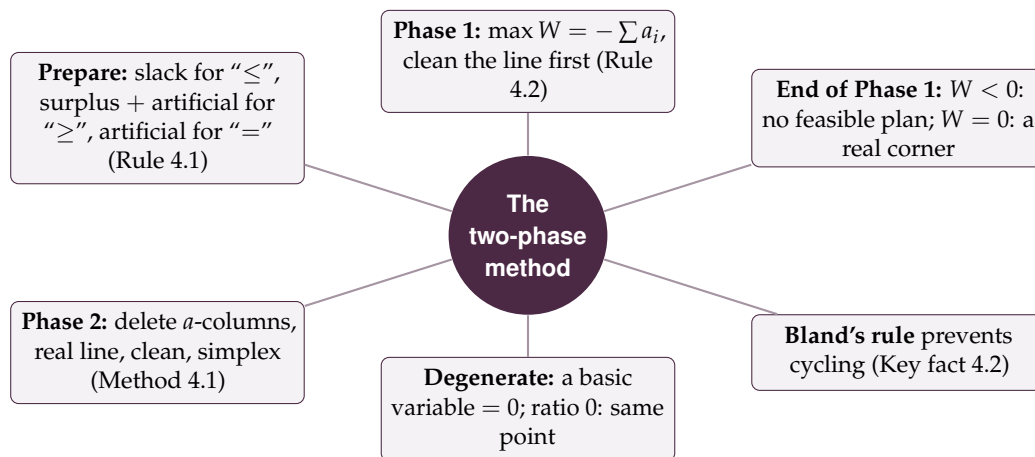
Part 6 in short

A degenerate corner has a basic variable equal to 0 (Definition 4.2); a ratio 0 gives a degenerate pivot, which changes the basis but not the point. It is not wasted, and Bland's rule (Key fact 4.2) guarantees that the method never turns in a circle. The complete simplex method, with its six signals, solves every linear program.

Check yourself

Q6. In the example of Section 20, choose x_2 as the first entering variable (it is allowed: its number -1 is negative). Run the simplex method. Is there a degenerate pivot?

Summary: the two-phase method in one picture



- ▶ When the origin is not feasible, prepare the equations with Rule 4.1: right-hand sides ≥ 0 ; a slack for " $\leq$ ", a surplus and an artificial variable for " $\geq$ ", an artificial variable for " $=$ ".
- ▶ An artificial variable measures what is missing in its line; a plan is real only when all are 0 (Definition 4.1).
- ▶ Phase 1: $\max W = -\sum a_i$, after cleaning the line (Rule 4.2). $W < 0$ at the end: no feasible plan (Key fact 4.1, Rule 4.3); $W = 0$: a real corner.
- ▶ Phase 2: delete the artificial columns, write and clean the real objective line, and run the simplex method (Method 4.1).
- ▶ Degeneracy: a basic variable equal to 0 (Definition 4.2); a ratio 0 gives a degenerate pivot (same point, new basis); Bland's rule prevents cycling (Key fact 4.2).
- ▶ The Big-M method of other books does the same work with a penalty M in the objective.

What comes next. The last z -line contains numbers like 25 and 5 (Lecture 3) or $\frac{20}{3}$ and $\frac{50}{3}$ (today). They prove optimality, and they also measure the value of one more unit of each resource. Their study is the subject of Chapter 3 (duality and sensitivity analysis).

Reading. F. S. Hillier and G. J. Lieberman, *Introduction to Operations Research*, Section 4.6; H. A. Taha, *Operations Research: An Introduction*, Sections 3.4 and 3.5.

Answers to “Check yourself”

Q1 (Part 1). $W = -a_1 - a_2$, so the raw line is $W + a_1 + a_2 = 0$. Clean it: $W \leftarrow W - L_1 - L_2$, with $L_1 = (1, 2, -1, 0, 1, 0 \mid 4)$ and $L_2 = (1, -1, 0, 0, 0, 1 \mid 1)$. The cleaned W -line is $(-2, -1, 1, 0, 0, 0 \mid -5)$ for the columns $x_1, x_2, e_1, s_3, a_1, a_2$; $-5 = -(4 + 1)$ is the value of W at the start. (Solving the problem gives the best plan $(2, 1)$ with cost 3.)

Q2 (Part 2). x_2 enters; the ratios are $8/1 = 8$ and $7/2 = \frac{7}{2}$, so a_2 leaves. The new point is $(0, \frac{7}{2})$: all the sensors are there, but $a_1 = 8 - \frac{7}{2} = \frac{9}{2}$ motors are missing, and $W = -\frac{9}{2}$. Then x_1 enters $(-\frac{3}{2})$; the ratios are 3 and 7, so a_1 leaves, and we reach $(3, 2)$ with $W = 0$: the same corner, after two pivots too.

Q3 (Part 3). Without cleaning, the line $(30, 40, 0, 0 \mid 0)$ has no negative number and its RHS is 0: we would conclude that the tableau is optimal with $z' = 0$, that is a cost of 0. This is false: the RHS of an objective line is the true value only when the basic columns have 0 in it.

Q4 (Part 4). The new line is $x_2 - e_4 + a_4 = 16$. In Phase 1, x_2 enters; the ratios are 20, 30, 18 and 16, so a_4 leaves and Phase 1 ends at $(0, 16)$ with $W = 0$. The old plan $B = (10, 15)$ makes only 15 drones B: it is not feasible any more. The feasible set is the part of the workshop with $16 \leq x_2 \leq 18$, and the best plan is the corner $(8, 16)$, where the assembly line meets the order line: $z = 40 \times 8 + 60 \times 16 = 1280$ thousand DA (check the corners $(8, 16)$: 1280; $(4, 18)$: 1240; $(0, 16)$: 960; $(0, 18)$: 1080). Here the order is binding: $e_4 = 0$.

Q5 (Part 5). The new line is $x_2 - e_4 + a_4 = 20$, and the cleaned W -line has -1 under x_2 and -20 on the right. x_2 enters; the ratios are 20, 30, 18 and 20, so s_3 leaves (cameras). Then the W -line has no negative number, and $W = -2$: there is no feasible plan (Rule 4.3). The last value $a_4 = 2$ means that even with all 18 cameras we miss 2 drones B: the order of 20 drones B cannot be met with 18 cameras.

Q6 (Part 6). x_2 enters; the line s_1 has -1 (no ratio), the ratio in L_2 is 4, so s_2 leaves: the corner $(0, 4)$ with $z = 4$. Then x_1 enters (-1) ; the ratios are $4/2 = 2$ and $4/1 = 4$, so s_1 leaves: $(2, 2)$ with $z = 6$, optimal. No ratio is 0, so there is no degenerate pivot: whether a degenerate pivot happens depends on the choices made.