

Lecture 3 — The Simplex Method

Moving from corner to corner with algebra



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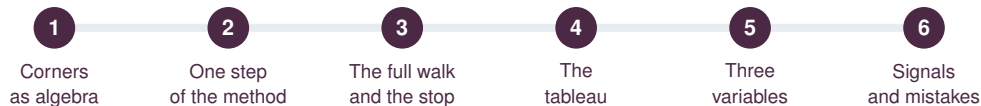
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Operations Research (O.R.)

What you will learn. In Lecture 2 we solved linear programs with two variables by drawing. Today we learn the *simplex method*: the algorithm that solves linear programs with any number of variables, without any drawing. At the end of this lecture you can:

- ▶ describe a corner of the feasible set with numbers (a *basic solution*);
- ▶ run the simplex method by hand, first with dictionaries, then with tableaux;
- ▶ read the best plan in the last tableau, and explain why it is the best;
- ▶ recognise two signals: an unbounded problem, and a problem with many optimal plans.

How this lecture is organised. The lecture has six parts, the same six parts as the slides, with the same titles. Each part starts with a banner that shows this map, so you always know where you are:



We keep the drone workshop of Lectures 1 and 2 as our example, so that every step of the algebra can be checked on the picture. Every substitution and every row operation is written in full. Definitions, rules, methods and key facts are in boxes. Each part ends with a short summary (“in short”) and a “Check yourself” question; the answers are at the end.

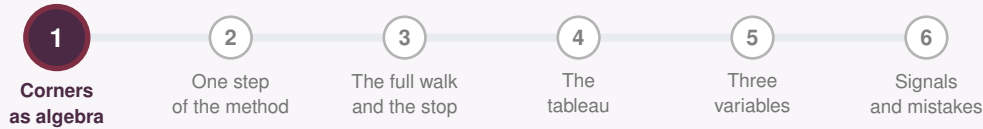
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Part 1 of 6

From corners to basic solutions



In this part, you learn to:

- ▶ describe a corner with numbers: which variables are 0;
- ▶ compute a basic solution and check if it is feasible;
- ▶ recognise two neighbouring corners.

1 Where we are: walk from corner to corner

In Lecture 2 we learned that if a linear program has a best plan, then a *corner* of the feasible set is optimal (Key fact 2.1). But checking all the corners is much too slow for large problems. With 50 variables and 100 constraints (150 constraints with the conditions $x_j \geq 0$), each corner is found by choosing 50 of the 150 constraints as equalities. There are $\binom{150}{50} \approx 2 \times 10^{40}$ such choices, so 2×10^{40} systems of equations to solve.

The better idea, seen at the end of Lecture 2, is to *walk*: start at a corner, move along an edge to a neighbouring corner with a better value, and stop when no neighbour is better (Figure 1). The *simplex method* does exactly this walk, but with algebra only, so it works for any number of variables. To do it, we must answer three questions with algebra:

1. How do we describe a corner without a drawing? (Part 1: basic solutions.)
2. How do we move to a better neighbouring corner? (Part 2: one step of the method.)
3. How do we know that we can stop? (Part 3: the optimality test.)

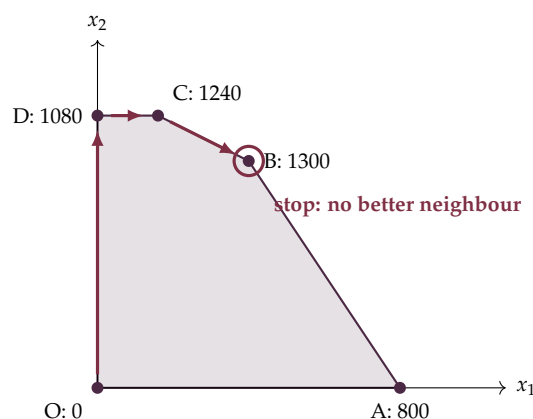


Figure 1: The walk on the drone workshop. Each corner shows its profit $z = 40x_1 + 60x_2$ (thousand DA). From O we go to D, then C, then B. At B, both neighbours (C and A) are worse, so we stop.

2 What we assume today

What we assume in this lecture

We solve problems of the form

$$\max z = c^T x \quad \text{subject to} \quad Ax \leq b, \quad x \geq 0, \quad \text{with } b \geq 0.$$

We maximise, all constraints are “ $\leq$ ”, and all right-hand sides are non-negative.

Why this assumption? We need a corner to start the walk. If $b \geq 0$, the origin $x = 0$ is feasible, because $A \cdot 0 = 0 \leq b$; and it is a corner (the axes meet there). So we can always start at the origin O (Figure 2, left).

A minimisation $\min z$ becomes $\max(-z)$ (Lecture 1, Rule 1.2), so minimisations are no problem. But with a constraint “ $\geq$ ” or “ $=$ ”, or with a negative right-hand side, the origin can be outside the feasible set (Figure 2, right). Then we need another way to find a first corner: this is the subject of Lecture 4.

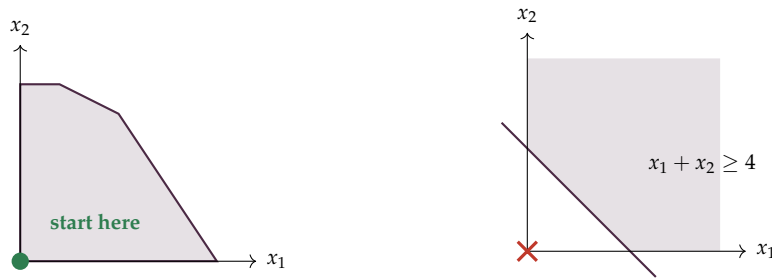


Figure 2: Left: in the drone workshop ($b \geq 0$) the origin O is feasible, so the walk can start there. Right: with the constraint $x_1 + x_2 \geq 4$, the origin is cut off (Lecture 4).

Our example. We use the drone workshop:

$$\begin{aligned} \max \quad & z = 40x_1 + 60x_2 && \text{(profit, thousand DA per week)} \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 40 && \text{(assembly hours)} \\ & 3x_1 + 2x_2 \leq 60 && \text{(testing hours)} \\ & x_2 \leq 18 && \text{(cameras)} \\ & x_1, x_2 \geq 0. \end{aligned}$$

Here $b = (40, 60, 18) \geq 0$. From Lecture 2 we know the answer: the corners are O (0,0), A (20,0), B (10,15), C (4,18), D (0,18), and the best one is B with $z = 1300$. Today we find this answer again without drawing, and we check every step on the picture.

3 Slack variables: the part we do not use

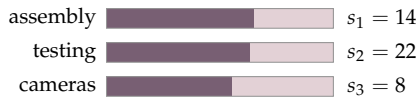
With one slack variable per constraint (Lecture 1, Rule 1.6), the drone workshop becomes

$$\begin{aligned} \max \quad & z = 40x_1 + 60x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + s_1 = 40 && (s_1 = \text{unused assembly hours}) \\ & 3x_1 + 2x_2 + s_2 = 60 && (s_2 = \text{unused testing hours}) \\ & x_2 + s_3 = 18 && (s_3 = \text{unused cameras}) \\ & x_1, x_2, s_1, s_2, s_3 \geq 0. \end{aligned}$$

Each slack is the part of a resource that the plan does *not* use. Two examples (Figure 3):

- The plan $P = (6, 10)$, inside the feasible set: assembly uses $6 + 2 \times 10 = 26$ of 40 hours, so $s_1 = 14$; testing uses $3 \times 6 + 2 \times 10 = 38$ of 60 hours, so $s_2 = 22$; cameras: 10 of 18, so $s_3 = 8$.
- The corner $B = (10, 15)$: assembly uses $10 + 30 = 40$ of 40 hours, so $s_1 = 0$; testing uses $30 + 30 = 60$ of 60 hours, so $s_2 = 0$; cameras: 15 of 18, so $s_3 = 3$.

Plan P = (6, 10): inside



Corner B = (10, 15)



Figure 3: Used part (dark) and unused part (light, the slack) of each resource. At the corner B, assembly and testing are fully used: $s_1 = s_2 = 0$.

Remember

A slack equal to 0 $\Leftrightarrow$ the resource is fully used $\Leftrightarrow$ the point is on the line of this constraint.

4 At a corner, two variables are zero

Look at the lines of the picture with this new eye (Figure 4). On the assembly line, $s_1 = 0$. On the testing line, $s_2 = 0$. On the camera line, $s_3 = 0$. On the vertical axis, $x_1 = 0$, and on the horizontal axis, $x_2 = 0$. So *each line of the picture is the place where one of the five variables is 0*.

A corner is a point where two lines meet. So at each corner, two of the five variables are 0:

- B is on the assembly line and the testing line: $s_1 = 0$ and $s_2 = 0$;
- C is on the assembly line and the camera line: $s_1 = 0$ and $s_3 = 0$;
- O is on the two axes: $x_1 = 0$ and $x_2 = 0$.

The standard form has 3 equations and 5 unknowns. When we know the 2 variables that are 0, there remain 3 equations with 3 unknowns, which usually have exactly one solution. This gives the idea of a basic solution.

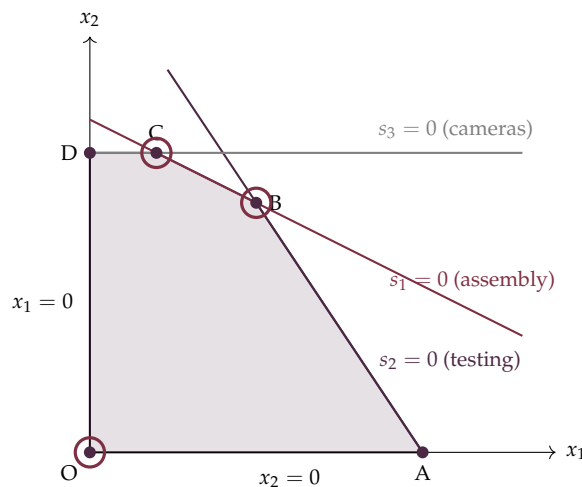


Figure 4: On each line, one variable is 0. The corners B, C and O (circled) are each on two lines, so two variables are 0 there.

5 Definition 3.1: basic solution

Definition 3.1 — Basic solution

A system in standard form has m equations. Choose m variables: the *basic* variables. The other variables are *non-basic*: put them equal to 0. Solve the equations for the basic variables. The result is a *basic solution*. If all its values are ≥ 0 , it is a *basic feasible solution* (BFS).

An easy way to remember: *basic* variables are the ones we solve for; *non-basic* variables are the ones we put at 0 (Figure 5). Here $m = 3$: we choose 3 basic variables, and the 2 others are non-basic.

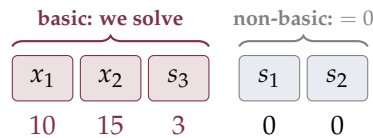


Figure 5: The basic solution with non-basic variables s_1, s_2 : we put $s_1 = s_2 = 0$ and solve for x_1, x_2, s_3 .

Example: non-basic s_1, s_2 .

- Put $s_1 = s_2 = 0$. The equations become $x_1 + 2x_2 = 40$, $3x_1 + 2x_2 = 60$, $x_2 + s_3 = 18$.
- The second equation minus the first gives $2x_1 = 20$, so $x_1 = 10$. Then $2x_2 = 40 - 10 = 30$, so $x_2 = 15$. The third equation gives $s_3 = 18 - 15 = 3$.
- The basic solution is $(x_1, x_2, s_1, s_2, s_3) = (10, 15, 0, 0, 3)$. All values are ≥ 0 : it is a basic feasible solution, and it is the corner B. (Assembly and testing are fully used; 3 cameras are left.)

6 Three choices, three results

Different choices of the non-basic variables give different results (Figure 6).

- Non-basic s_1, s_2 .** The assembly line and the testing line meet at $(10, 15)$, and $s_3 = 3 \geq 0$.
Feasible: the corner B ✓
- Non-basic x_1, s_1 .** With $x_1 = s_1 = 0$, the first equation gives $2x_2 = 40$, so $x_2 = 20$. Then $s_2 = 60 - 40 = 20$ and $s_3 = 18 - 20 = -2$. The basic solution $(0, 20, 0, 20, -2)$ has a negative value: **not feasible ✗**. In words: 20 drones B would need 20 cameras, but there are only 18. On the picture, it is the point $(0, 20)$ where the vertical axis meets the assembly line, outside the feasible set.
- Non-basic x_2, s_3 .** Then $x_2 = 0$ and the camera equation $x_2 + s_3 = 18$ gives $0 + 0 = 18$: impossible. On the picture, the horizontal axis ($x_2 = 0$) and the camera line ($x_2 = 18$) are parallel: they never meet. **No solution.**

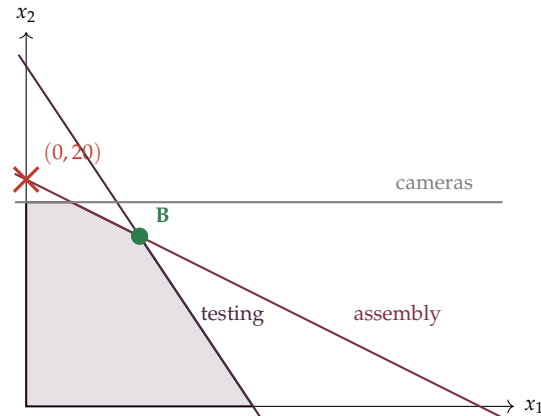


Figure 6: Choice 1: the assembly and testing lines meet at B, inside: feasible. Choice 2: the vertical axis and the assembly line meet at $(0, 20)$, outside: not feasible. Choice 3: the horizontal axis and the camera line are parallel: no point.

All the choices. There are $\binom{5}{2} = 10$ ways to choose the 2 non-basic variables. The table below gives all of them, and Figure 7 shows the points.

Non-basic (= 0)	Basic solution $(x_1, x_2, s_1, s_2, s_3)$	Result
x_1, x_2	$(0, 0, 40, 60, 18)$	feasible: corner O
x_2, s_2	$(20, 0, 20, 0, 18)$	feasible: corner A
s_1, s_2	$(10, 15, 0, 0, 3)$	feasible: corner B
s_1, s_3	$(4, 18, 0, 12, 0)$	feasible: corner C
x_1, s_3	$(0, 18, 4, 24, 0)$	feasible: corner D
x_1, s_1	$(0, 20, 0, 20, -2)$	not feasible
x_1, s_2	$(0, 30, -20, 0, -12)$	not feasible
x_2, s_1	$(40, 0, 0, -60, 18)$	not feasible
s_2, s_3	$(8, 18, -4, 0, 0)$	not feasible
x_2, s_3	no solution (parallel lines)	—

So there are 9 basic solutions, and the 5 feasible ones are exactly the 5 corners. This is true with any number of variables:

Key fact 3.1

The basic feasible solutions are exactly the corners of the feasible set.

Why? Putting a variable equal to 0 means being on a line. So choosing 2 non-basic variables means choosing 2 lines and taking the point where they meet. If this point satisfies all the constraints (all values ≥ 0), it is a corner; otherwise it is outside. This is exactly Method 2.4 of Lecture 2, written with the slack variables. From now on, “corner” and “basic feasible solution” mean the same thing, and we can describe a corner without a drawing: we just say which variables are 0.

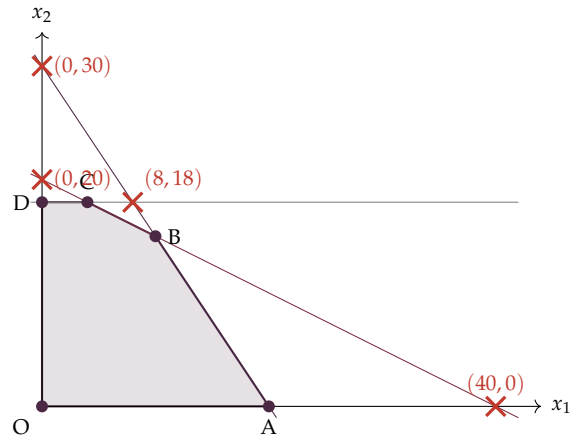


Figure 7: The 9 basic solutions of the drone workshop: the 5 corners (dots) are feasible; the 4 other points (crosses) are outside.

7 Neighbouring corners: one in, one out

Each corner has its pair of non-basic variables, its “zeros” (Figure 8). Two corners joined by an edge are called *neighbours*, and their pairs share *one* variable. For example:

- ▶ O has $\{x_1, x_2\}$ and D has $\{x_1, s_3\}$. They share x_1 : the edge from O to D is on the vertical axis, where $x_1 = 0$. Moving from O to D, x_2 stops being 0 (we say that x_2 *enters* the basis) and s_3 becomes 0 (we say that s_3 *leaves* the basis).
- ▶ C has $\{s_1, s_3\}$ and B has $\{s_1, s_2\}$. They share s_1 : the edge from C to B is on the assembly line. Moving from C to B, s_3 enters and s_2 leaves.

So *one step of the simplex method is one exchange: one variable enters, and one variable leaves*. This exchange is called a *pivot*. Part 2 explains how to choose the two variables.

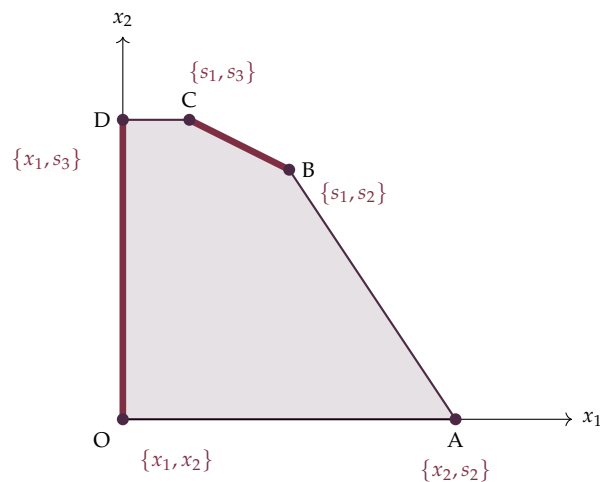


Figure 8: Each corner with its pair of non-basic variables. The two thick edges are the moves $O \rightarrow D$ and $C \rightarrow B$: neighbouring corners share one non-basic variable.

Part 1 in short

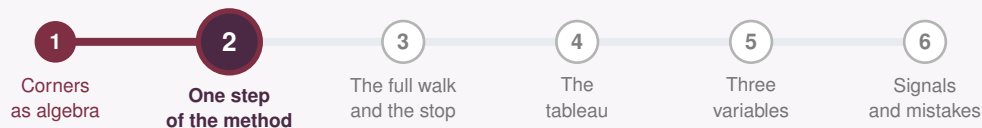
A slack is the unused part of a resource. A corner is a point where two lines meet, so two variables are 0 there. A *basic solution* puts m variables aside as basic (we solve for them) and the others at 0; the feasible ones are exactly the corners (Key fact 3.1). Moving to a neighbouring corner = one variable enters, one variable leaves.

Check yourself

Q1. Compute the basic solution with non-basic variables x_2 and s_2 . Is it feasible? Which corner is it?

Part 2 of 6

One step of the simplex method



In this part, you learn to:

- ▶ write the dictionary of a corner;
- ▶ choose the variable that enters (who grows?);
- ▶ do the ratio test (how far?) and the pivot (the new corner).

8 The dictionary of the corner O

To move from corner to corner, we must see how the basic variables and the profit change when a non-basic variable changes. A dictionary shows exactly this.

Definition 3.2 — Dictionary

A *dictionary* writes each basic variable, and the objective z , as a function of the non-basic variables only.

At the start, the basic variables are the slacks, and the non-basic variables are x_1 and x_2 . We isolate each slack in its equation:

$$\begin{array}{rclcl}
 x_1 + 2x_2 + s_1 = 40 & \Rightarrow & s_1 = 40 - x_1 - 2x_2 \\
 3x_1 + 2x_2 + s_2 = 60 & \Rightarrow & s_2 = 60 - 3x_1 - 2x_2 \\
 x_2 + s_3 = 18 & \Rightarrow & s_3 = 18 - x_2 \\
 \hline
 z = 40x_1 + 60x_2 & \Rightarrow & z = 0 + 40x_1 + 60x_2
 \end{array}$$

This is the same system as the standard form, only written differently: every plan that satisfies the equations of the standard form satisfies the dictionary, and conversely.

Reading a dictionary. Put the non-basic variables to 0 and read the values on the left: here $x_1 = x_2 = 0$, so $s_1 = 40$, $s_2 = 60$, $s_3 = 18$, and $z = 0$. This is the corner O: nothing is made, nothing is used, profit 0. The constants of the dictionary (40, 60, 18 and 0) are the values at the current corner.

What else does it tell us? The coefficients show what happens if a non-basic variable becomes positive. The line $z = 0 + 40x_1 + 60x_2$ says that each drone A adds 40 to the profit and each drone B adds 60. The line $s_3 = 18 - x_2$ says that each drone B uses one of the 18 cameras.

9 Question 1: who enters?

At the corner O, both x_1 and x_2 are 0. In $z = 0 + 40x_1 + 60x_2$:

- ▶ one more drone A (increase x_1 by 1, keep $x_2 = 0$) increases z by 40;
- ▶ one more drone B (increase x_2 by 1, keep $x_1 = 0$) increases z by 60.

These are the two directions in which we can leave O: along the x_1 -axis or along the x_2 -axis (Figure 9). A non-basic variable with a *positive* coefficient in the z -line can improve the plan. With a negative coefficient it would lower z , and with a coefficient 0 it would not change z .

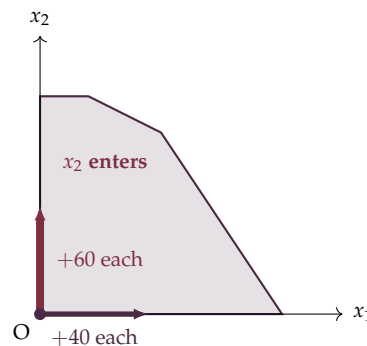


Figure 9: From O we can make drones A (+40 each) or drones B (+60 each). With the largest number, x_2 enters.

Rule 3.1 — Entering variable

Choose a non-basic variable with a positive coefficient in the z -line of the dictionary. Usually we take the largest coefficient; if there is a tie, the smallest index. If no coefficient is positive, the current corner is optimal (Key fact 3.2).

Here the largest coefficient is 60: x_2 **enters** (we start making drones B). Choosing x_1 would also be correct, because its coefficient is positive too: any positive coefficient improves the plan. The “largest coefficient” rule is only a simple habit, and it is not always the fastest (Check yourself Q2, and TD 3).

10 Question 2: how far can x_2 grow? (ratio test)

We increase x_2 and keep the other non-basic variable at 0: $x_1 = 0$ and $x_2 = t$ with $t \geq 0$. The dictionary gives the basic variables, which must stay ≥ 0 . Let us try some values of t :

t	$s_1 = 40 - 2t$	$s_2 = 60 - 2t$	$s_3 = 18 - t$	
0	40	60	18	start: the corner O
10	20	40	8	all ≥ 0 : inside the feasible set
18	4	24	0	all the cameras are used: the corner D
20	0	20	-2	too far: 20 cameras needed, only 18

Each line of the dictionary gives a limit:

- ▶ $s_1 = 40 - 2t \geq 0$, so $t \leq 40/2 = 20$;
- ▶ $s_2 = 60 - 2t \geq 0$, so $t \leq 60/2 = 30$;
- ▶ $s_3 = 18 - t \geq 0$, so $t \leq 18/1 = 18$.

All three conditions must hold, so $t \leq 18$, the *smallest* of the three numbers. These numbers are called *ratios*: (constant of the line) / (coefficient of x_2 , without its minus sign). At $t = 18$, s_3 becomes 0: s_3 **leaves**.

The picture (Figure 10). We move up the x_2 -axis from O. The three constraint lines cross this axis at $x_2 = 18$ (cameras), 20 (assembly) and 30 (testing). The first line we meet stops us: *the smallest ratio is the first wall on our way*.

Which lines give a limit? Only the lines where the entering variable has a *negative* coefficient in the dictionary: then the basic variable decreases when t grows. If the coefficient is positive, the basic variable grows with t ; if it is 0, it does not change. In both cases it never becomes negative, so the line gives no limit.

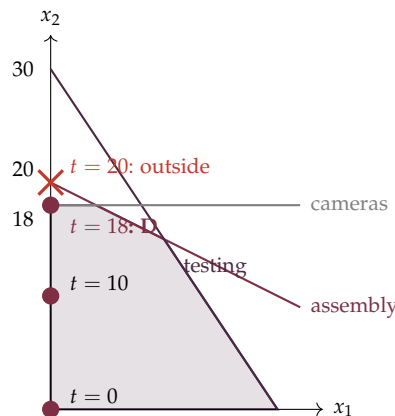


Figure 10: The ratio test for x_2 : moving up from O, we are inside at $t = 0$ and $t = 10$; the first wall is the camera line at $t = 18$ (the corner D); at $t = 20$ we would be outside.

11 Question 3: the pivot (the new dictionary)

We now write the new dictionary, where x_2 is basic and s_3 is non-basic.

1. From the line of the leaving variable, $s_3 = 18 - x_2$, we isolate the entering variable:

$$x_2 = 18 - s_3.$$

2. We replace x_2 by $18 - s_3$ in all the other lines:

$$\begin{aligned} s_1 &= 40 - x_1 - 2(18 - s_3) = 40 - x_1 - 36 + 2s_3 = 4 - x_1 + 2s_3 \\ s_2 &= 60 - 3x_1 - 2(18 - s_3) = 60 - 3x_1 - 36 + 2s_3 = 24 - 3x_1 + 2s_3 \\ z &= 40x_1 + 60(18 - s_3) = 1080 + 40x_1 - 60s_3. \end{aligned}$$

3. The new dictionary, with non-basic variables x_1 and s_3 , is

$$\begin{array}{rcl} x_2 & = & 18 - s_3 \\ s_1 & = & 4 - x_1 + 2s_3 \\ s_2 & = & 24 - 3x_1 + 2s_3 \\ \hline z & = & 1080 + 40x_1 - 60s_3. \end{array}$$

Why is this correct? We only solved one equation for x_2 and substituted. This never changes the solutions of a system. So the new dictionary describes exactly the same plans as before; only the variables on the right have changed.

Reading. With $x_1 = s_3 = 0$: $x_2 = 18$, $s_1 = 4$, $s_2 = 24$, $z = 1080$. This is the corner $D = (0, 18)$: 18 drones B, all cameras used, 4 assembly hours and 24 testing hours left. The profit went from 0 to 1080 thousand DA. The new constants are all ≥ 0 : the ratio test guarantees it, because we stopped at the first wall.

12 One step of the simplex method = three questions

One step of the simplex method is always the same three questions (Figure 11):

1. **Who enters?** A non-basic variable with a positive number in the z-line (usually the largest).
2. **How far? Who leaves?** The ratio test: the smallest ratio, which is the first wall we meet. The basic variable of this line leaves.
3. **Pivot.** Rewrite the dictionary: we get a new corner, with a better z .

We repeat until there is no positive number in the z-line: then the corner is optimal (Part 3). Our first step was: x_2 enters, s_3 leaves (ratio 18), and we arrive at the corner D, with z going from 0 to 1080.

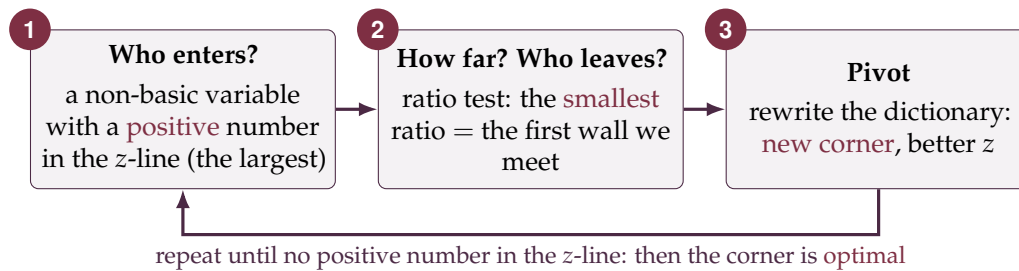


Figure 11: One step of the simplex method.

Part 2 in short

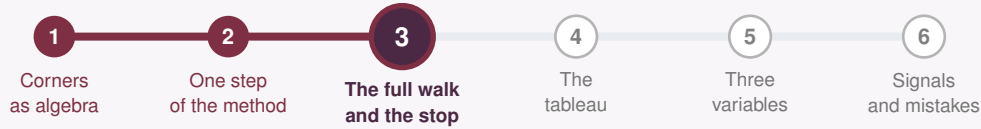
A dictionary writes the basic variables and z with the non-basic variables; we read the corner by putting the non-basic variables at 0. One step: a variable with a positive number in the z-line enters (Rule 3.1); the smallest ratio, the first wall, tells who leaves; the pivot (isolate, then substitute) gives the new dictionary of a better corner.

Check yourself

Q2. In the first dictionary, choose x_1 as entering variable instead of x_2 . Do the ratio test: which variable leaves, and which corner do you reach?

Part 3 of 6

The full walk, and when to stop



In this part, you learn to:

- ▶ finish the drone workshop: $D \rightarrow C \rightarrow B$;
- ▶ understand a surprising step: a slack enters;
- ▶ explain why we can stop: the last z -line is a proof.

We repeat the three questions until no number of the z -line is positive.

13 Second step: x_1 enters

The dictionary at D is

$$x_2 = 18 - s_3, \quad s_1 = 4 - x_1 + 2s_3, \quad s_2 = 24 - 3x_1 + 2s_3, \quad z = 1080 + 40x_1 - 60s_3.$$

1. Who enters? In the z -line only x_1 has a positive coefficient: x_1 **enters**. (The -60 says that leaving cameras unused would now lower the profit: each unused camera means one drone B less.)

2. How far? We keep $s_3 = 0$ and put $x_1 = t$:

- ▶ $s_1 = 4 - t \geq 0$, so $t \leq 4$;
- ▶ $s_2 = 24 - 3t \geq 0$, so $t \leq 8$;
- ▶ $x_2 = 18 - s_3$ does not contain x_1 : no limit.

The smallest ratio is 4: s_1 **leaves**. On the picture (Figure 12) we move right along the camera line from D, and we meet the assembly line at $x_1 = 4$ before the testing line at $x_1 = 8$.

3. Pivot. From $s_1 = 4 - x_1 + 2s_3$ we get $x_1 = 4 - s_1 + 2s_3$. We substitute:

$$x_1 = 4 - s_1 + 2s_3$$

$$s_2 = 24 - 3(4 - s_1 + 2s_3) + 2s_3 = 24 - 12 + 3s_1 - 6s_3 + 2s_3 = 12 + 3s_1 - 4s_3$$

$$x_2 = 18 - s_3$$

$$z = 1080 + 40(4 - s_1 + 2s_3) - 60s_3 = 1080 + 160 - 40s_1 + 80s_3 - 60s_3 = 1240 - 40s_1 + 20s_3.$$

Reading ($s_1 = s_3 = 0$): $x_1 = 4$, $x_2 = 18$, $s_2 = 12$, $z = 1240$: the corner $C = (4, 18)$.

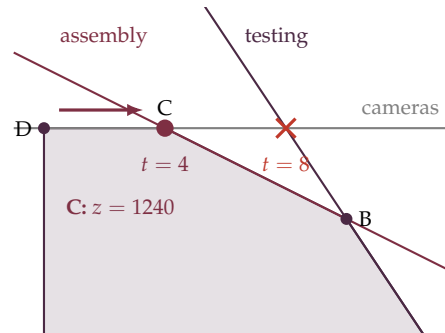


Figure 12: Zoom on the top of the picture. From D we move right along the camera line. The first wall is the assembly line at $t = 4$ (the corner C); the testing line comes later, at $t = 8$.

14 Third step: a surprise — s_3 enters

The dictionary at C is

$$x_1 = 4 - s_1 + 2s_3, \quad s_2 = 12 + 3s_1 - 4s_3, \quad x_2 = 18 - s_3, \quad z = 1240 - 40s_1 + 20s_3.$$

1. Who enters? The coefficient of s_3 is positive: s_3 **enters**. A slack can enter too! But s_3 is the number of *unused cameras*. Why would we leave cameras unused? The dictionary explains it (Figure 13). Keep the assembly fully used ($s_1 = 0$) and leave one camera unused ($s_3 = 1$):

- ▶ the line of x_2 gives $x_2 = 18 - 1$: one drone B less, so -60 ; this frees 2 assembly hours;
- ▶ the line of x_1 gives $x_1 = 4 + 2$: with the 2 free hours we make two drones A more, so $+80$;
- ▶ in total, $-60 + 80 = +20$ for each unused camera: exactly the 20 in the z -line.

So each number of the dictionary has a meaning. Here drones A use the assembly hours better, so we trade drones B for drones A.

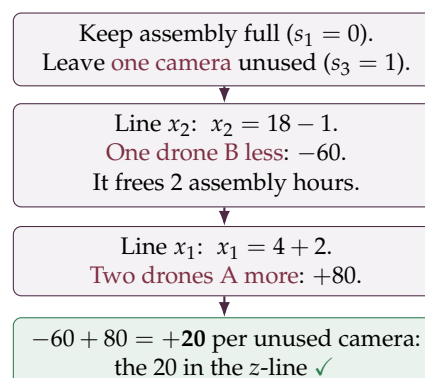


Figure 13: The meaning of the $+20$ in the z -line at the corner C.

2. How far? We keep $s_1 = 0$ and put $s_3 = t$:

- ▶ $x_1 = 4 + 2t$ grows: no limit;
- ▶ $s_2 = 12 - 4t \geq 0$, so $t \leq 12/4 = 3$;
- ▶ $x_2 = 18 - t \geq 0$, so $t \leq 18$.

The smallest ratio is 3: s_2 **leaves**. On the picture (Figure 14) we slide along the assembly line from C: at $t = 1$ and $t = 2$ we are still inside; at $t = 3$ we reach the testing line, at B; at $t = 4$ we would be outside, with $s_2 = -4$.

3. Pivot. From $s_2 = 12 + 3s_1 - 4s_3$: $4s_3 = 12 + 3s_1 - s_2$, so $s_3 = 3 + \frac{3}{4}s_1 - \frac{1}{4}s_2$. We substitute:

$$x_1 = 4 - s_1 + 2\left(3 + \frac{3}{4}s_1 - \frac{1}{4}s_2\right) = 10 + \frac{1}{2}s_1 - \frac{1}{2}s_2$$

$$s_3 = 3 + \frac{3}{4}s_1 - \frac{1}{4}s_2$$

$$x_2 = 18 - \left(3 + \frac{3}{4}s_1 - \frac{1}{4}s_2\right) = 15 - \frac{3}{4}s_1 + \frac{1}{4}s_2$$

$$z = 1240 - 40s_1 + 20\left(3 + \frac{3}{4}s_1 - \frac{1}{4}s_2\right) = 1300 - 25s_1 - 5s_2.$$

Reading ($s_1 = s_2 = 0$): $x_1 = 10$, $x_2 = 15$, $s_3 = 3$, $z = 1300$: the corner B = (10, 15).

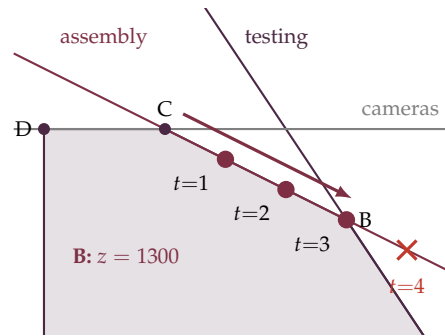


Figure 14: Zoom on the top of the picture. From C we slide along the assembly line; the first wall is the testing line at $t = 3$: the corner B.

15 Stop: B is optimal — and here is the proof

The last dictionary has $z = 1300 - 25s_1 - 5s_2$. No coefficient is positive: increasing s_1 or s_2 can only lower z . So no neighbour of B is better, and we stop.

Why is B the best of *all* plans, and not only better than its neighbours? Here is the argument, step by step.

1. The dictionary is just the original equations rewritten. So the equation $z = 1300 - 25s_1 - 5s_2$ is true for *every* plan that satisfies the equations, not only near B.
2. For every feasible plan, $s_1 \geq 0$ and $s_2 \geq 0$ (they are unused hours).
3. Therefore $z = 1300 - (\text{something} \geq 0) - (\text{something} \geq 0) \leq 1300$ for every feasible plan.
4. B gives exactly 1300. So B is optimal.

On the picture (Figure 15), the profit line $40x_1 + 60x_2 = 1300$ touches the feasible set only at B; above it there is no feasible plan.

Key fact 3.2 — Optimality

If, in a dictionary, no non-basic variable has a positive coefficient in the z -line, then the current corner is optimal. Its value is the constant of the z -line.

The numbers 25 and 5. In Lecture 2 (“how can we be sure?”) we proved the optimality of B by adding $25 \times$ (assembly constraint) + $5 \times$ (testing constraint). The simplex method gives these numbers for free: they are the coefficients of s_1 and s_2 in the last z -line. They also have a meaning: one more assembly hour would increase the best profit by 25 thousand DA, and one more testing hour by 5 thousand DA (for small changes; this is studied in Chapter 3, sensitivity analysis).

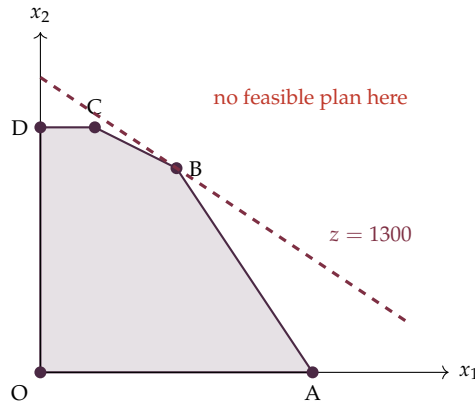


Figure 15: The profit line $z = 1300$ touches the feasible set only at B.

16 The whole walk, and Method 3.1

Here is the whole run:

Step	Enters	Leaves	Ratio	Corner	z
start	—	—	—	O (0,0)	0
1	x_2	s_3	18	D (0,18)	1080
2	x_1	s_1	4	C (4,18)	1240
3	s_3	s_2	3	B (10,15)	1300

Each step follows an edge to a better neighbour: exactly the walk of Figure 1. We visited 4 of the 5 corners, without any drawing. With other choices the path can change: if x_1 enters first, the path is $O \rightarrow A \rightarrow B$ (Check yourself Q2 and TD 2, Exercise 4). We can now write the whole method; Figure 16 shows it as a picture.

Method 3.1 — The simplex method with dictionaries

1. **Start:** the basic variables are the slacks; the corner is O (this needs $b \geq 0$).
2. **Who enters?** A non-basic variable with a positive coefficient in the z -line (Rule 3.1). If there is none, **stop:** the corner is optimal (Key fact 3.2).
3. **How far?** For each line where the entering variable has a *negative* coefficient, compute (constant) / (size of the coefficient). The smallest ratio gives the leaving variable. Lines where the coefficient is ≥ 0 give no limit. (No limit at all: unbounded, Part 6.)
4. **Pivot:** solve the line of the leaving variable for the entering variable, and substitute it in all the other lines and in z .
5. **Read** the new corner (non-basic variables = 0) and go back to step 2.

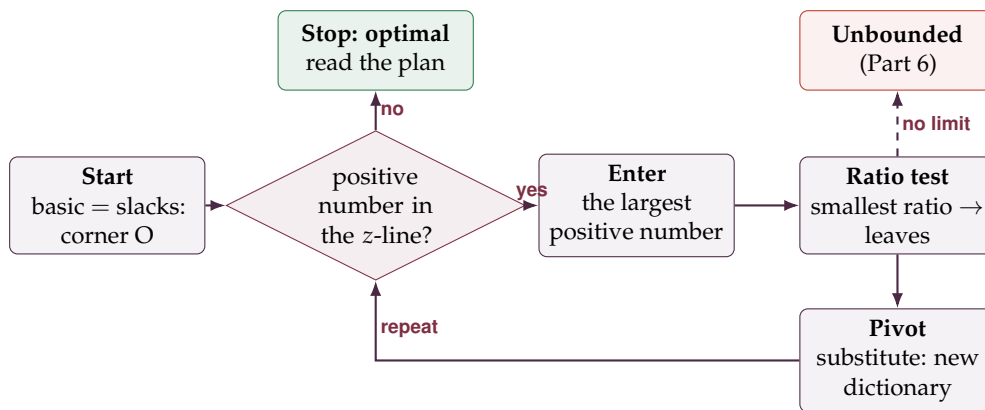


Figure 16: Method 3.1 as a picture.

Part 3 in short

The walk $O \rightarrow D \rightarrow C \rightarrow B$ uses the same three questions at each step. A slack can enter: it means “leave this resource partly unused”, and the number in the z-line says why it pays. We stop when no number of the z-line is positive; then the last z-line, $z = 1300 - 25s_1 - 5s_2$, is a complete proof that no feasible plan is better (Key fact 3.2).

Check yourself

Q3. Use the last dictionary of the drone workshop. If the workshop decides to keep one assembly hour unused ($s_1 = 1, s_2 = 0$), what are x_1, x_2 and z ? Check the value of z with $40x_1 + 60x_2$.

Part 4 of 6

The simplex tableau



In this part, you learn to:

- ▶ write the same equations in a table: the tableau;
- ▶ find the pivot in the table;
- ▶ do the row operations and read the new corner.

17 From the dictionary to the tableau

A dictionary is long to write when there are many variables, and the substitutions are easy to get wrong. In practice we write the same equations in a table, the *simplex tableau*, and we replace the substitutions by row operations.

In the tableau, each equation is written with all the variables on the left. The constraints are already in this form; the objective $z = 40x_1 + 60x_2$ is written

$$z - 40x_1 - 60x_2 = 0.$$

We only write the coefficients (Figure 17).

dictionary				tableau						
$s_1 = 40 - x_1 - 2x_2$		variables to the left →		T0	x_1	x_2	s_1	s_2	s_3	RHS
$s_2 = 60 - 3x_1 - 2x_2$				s_1	1	2	1	0	0	40
$s_3 = 18 - x_2$				s_2	3	2	0	1	0	60
$z = 0 + 40x_1 + 60x_2$				s_3	0	1	0	0	1	18
				z	-40	-60	0	0	0	0

Figure 17: The same equations as a dictionary (left) and as the first tableau T0 (right). The unit columns of the basic variables are shaded, and the RHS column (green) gives their values.

How to read a tableau.

- ▶ Each line is an equation. The line s_1 reads $x_1 + 2x_2 + s_1 = 40$; the line z reads $z - 40x_1 - 60x_2 = 0$. (RHS means “right-hand side”.)
- ▶ The first column gives the basic variable of each line. The columns of the basic variables are *unit columns*: a 1 in their own line and 0 everywhere else, also in the z -line.
- ▶ The RHS column gives the values of the basic variables at the current corner, and the last RHS gives the value of z .
- ▶ In the z -line, the signs are *opposite* to the dictionary, because we moved the terms to the left. So a *negative* number in the z -line of the tableau means that z grows when this variable grows.

18 Rule 3.2: the three questions in the tableau

The three questions are the same; only the way of writing changes:

	Dictionary	Tableau
1. Who enters?	the largest positive number in the z-line	the most negative number in the z-line
2. How far?	lines with a negative coefficient: constant $\div$ size of the coefficient	lines with a positive number: RHS $\div$ this number; the smallest ratio
3. Pivot	solve the leaving line, substitute	divide the pivot line by the pivot; make 0 in the rest of the column (row operations)
Stop when	no positive number in the z-line	no negative number in the z-line

Rule 3.2 — Simplex tableau

1. **Entering column:** the most negative number in the z-line. If there is no negative number, the tableau is **optimal**.
2. **Ratio test:** for each line with a *positive* number in the entering column, compute RHS / (this number). The smallest ratio gives the *pivot line*; its basic variable leaves. Lines with 0 or a negative number are not used.
3. **Pivot:** divide the pivot line by the pivot number (so the pivot becomes 1); then, with row operations, put 0 everywhere else in the entering column, the z-line included. Write the entering variable as the basic variable of the pivot line.

Why “positive” here, and “negative” in the dictionary? The line $s_1 = 40 - x_1 - 2x_2$ of the dictionary is the line $x_1 + 2x_2 + s_1 = 40$ of the tableau: the coefficients of the non-basic variables change sign when we move them to the left. So the rule is the same.

Why row operations? In a dictionary we solved one equation for the entering variable and substituted it in the others. In a tableau we do the same thing with two operations on the lines: *divide a line by a non-zero number*, and *add to a line a multiple of another line*. These operations never change the solutions of a system (this is the Gauss method of linear algebra). The goal of a pivot is to make the column of the entering variable a unit column, with the 1 in the pivot line: then this variable is basic in this line, exactly as in the new dictionary.

19 The three pivots in the tableau

We write L_1, L_2, L_3 for the three lines and z for the last line. In each figure, the old tableau is on top: the entering column is shaded, the ratios are on the right, the pivot and the smallest ratio are boxed, and the leaving line is shaded. The new tableau is below, with the row operation of each line on its right (L_i always means the *new* pivot line).

Pivot 1: corner O $\rightarrow$ D (Figure 18). The most negative number in the z-line is -60 : x_2 enters. Ratios: $40/2 = 20$, $60/2 = 30$, $18/1 = 18$. The smallest is 18: the pivot line is L_3 , and s_3 leaves. Each row operation is chosen to make a 0 in the column x_2 . For example:

$$\begin{aligned}\text{new } L_1 &= (1, 2, 1, 0, 0 \mid 40) - 2 \times (0, 1, 0, 0, 1 \mid 18) = (1, 0, 1, 0, -2 \mid 4), \\ \text{new } z &= (-40, -60, 0, 0, 0 \mid 0) + 60 \times (0, 1, 0, 0, 1 \mid 18) = (-40, 0, 0, 0, 60 \mid 1080).\end{aligned}$$

Tableau T1 is the corner D ($x_2 = 18$, $s_1 = 4$, $s_2 = 24$, $z = 1080$). Compare with the dictionary of

Section 11: the line s_1 of T1, $x_1 + s_1 - 2s_3 = 4$, is the line $s_1 = 4 - x_1 + 2s_3$ of the dictionary, and the z-line, $z - 40x_1 + 60s_3 = 1080$, is $z = 1080 + 40x_1 - 60s_3$.

		enters ↓					
T0	x_1	x_2	s_1	s_2	s_3	RHS	ratio
s_1	1	2	1	0	0	40	$40/2 = 20$
s_2	3	2	0	1	0	60	$60/2 = 30$
leaves s_3	0	1	0	0	1	18	$18/1 = 18$
z	-40	-60	0	0	0	0	
pivot ↓							
T1	x_1	x_2	s_1	s_2	s_3	RHS	
s_1	1	0	1	0	-2	4	$L_1 - 2L_3$
s_2	3	0	0	1	-2	24	$L_2 - 2L_3$
x_2	0	1	0	0	1	18	$L_3 \div 1$
z	-40	0	0	0	60	1080	$z + 60L_3$

Figure 18: Pivot 1: x_2 enters, s_3 leaves. T1 is the corner D.

Pivot 2: corner D $\rightarrow$ C (Figure 19). The only negative number in the z-line is -40: x_1 enters. Ratios: $4/1 = 4$, $24/3 = 8$; the line x_2 has 0 in this column, so no ratio. The pivot line is L_1 ; s_1 leaves. Row operations: L_1 is divided by 1 (no change); $L_2 \leftarrow L_2 - 3L_1$; L_3 is unchanged (it already has 0); $z \leftarrow z + 40L_1$. Tableau T2 is the corner C ($x_1 = 4$, $x_2 = 18$, $s_2 = 12$, $z = 1240$). The z-line still has a negative number, -20: we continue.

		enters ↓					
T1	x_1	x_2	s_1	s_2	s_3	RHS	ratio
leaves s_1	1	0	1	0	-2	4	$4/1 = 4$
s_2	3	0	0	1	-2	24	$24/3 = 8$
x_2	0	1	0	0	1	18	0: no ratio
z	-40	0	0	0	60	1080	
pivot ↓							
T2	x_1	x_2	s_1	s_2	s_3	RHS	
x_1	1	0	1	0	-2	4	$L_1 \div 1$
s_2	0	0	-3	1	4	12	$L_2 - 3L_1$
x_2	0	1	0	0	1	18	unchanged
z	0	0	40	0	-20	1240	$z + 40L_1$

Figure 19: Pivot 2: x_1 enters, s_1 leaves. T2 is the corner C.

Pivot 3: corner C $\rightarrow$ B (Figure 20). The z-line has -20 in the column s_3 : s_3 enters. Ratios: the line x_1 has -2 (negative): no ratio; $12/4 = 3$; $18/1 = 18$. The pivot is the 4 in L_2 ; s_2 leaves. Row operations: first $L_2 \leftarrow L_2/4 = (0, 0, -\frac{3}{4}, \frac{1}{4}, 1 | 3)$; then $L_1 \leftarrow L_1 + 2L_2$, $L_3 \leftarrow L_3 - L_2$, $z \leftarrow z + 20L_2$. For example the new z-line is

$$(0, 0, 40, 0, -20 | 1240) + 20 \times (0, 0, -\frac{3}{4}, \frac{1}{4}, 1 | 3) = (0, 0, 25, 5, 0 | 1300).$$

There is no negative number in the z-line of T3: it is **optimal**. We read $x_1 = 10$, $x_2 = 15$, $s_3 = 3$, $s_1 = s_2 = 0$, $z = 1300$. The z-line reads $z + 25s_1 + 5s_2 = 1300$, that is $z = 1300 - 25s_1 - 5s_2$: the same as the last dictionary.

enters ↓

T2	x_1	x_2	s_1	s_2	s_3	RHS	ratio
x_1	1	0	1	0	-2	4	-2: no ratio
s_2	0	0	-3	1	4	12	12/4 = 3
x_2	0	1	0	0	1	18	18/1 = 18
z	0	0	40	0	-20	1240	

leaves

↓ pivot

T3	x_1	x_2	s_1	s_2	s_3	RHS	
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	10	$L_1 + 2L_2$
s_3	0	0	$-\frac{3}{4}$	$\frac{1}{4}$	1	3	$L_2 \div 4$
x_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	15	$L_3 - L_2$
z	0	0	25	5	0	1300	$z + 20L_2$

Figure 20: Pivot 3: s_3 enters, s_2 leaves. T3 is the corner B, and it is optimal.

20 Two automatic checks, and Method 3.2

After each pivot, two things must be true. If one is false, there is a mistake.

- ▶ **The RHS column stays ≥ 0 .** Take a line with a positive number a in the entering column and RHS r , while the pivot line has pivot p and RHS r_p . The row operation gives the new RHS $r - a \times \frac{r_p}{p}$. Since $\frac{r_p}{p}$ is the smallest ratio, $\frac{r_p}{p} \leq \frac{r}{a}$, so $a \times \frac{r_p}{p} \leq r$, and the new RHS is ≥ 0 . For a line with $a \leq 0$, the RHS can only grow. (Example, pivot 3: the line x_2 becomes $18 - 1 \times 3 = 15 \geq 0$.)
- ▶ **The value z never decreases.** The new value of z is the old value plus (size of the entering number) $\times$ (smallest ratio): $0 + 60 \times 18 = 1080$, then $1080 + 40 \times 4 = 1240$, then $1240 + 20 \times 3 = 1300$.

Also check that the columns of the basic variables are unit columns.

Method 3.2 — The simplex method with tableaux

1. Write the standard form with slack variables. Build the first tableau: the basic variables are the slacks, and the z-line contains $-c$ (minus the coefficients of the objective).
2. Find the most negative number in the z-line: the entering column. If there is none, **stop**: the tableau is optimal.
3. Compute the ratios RHS / (positive numbers of the entering column); the smallest gives the pivot line. (No positive number: unbounded, Part 6.)
4. Divide the pivot line by the pivot; make 0 elsewhere in the column with row operations (the z-line included).
5. Read the new corner: basic variables = RHS, non-basic variables = 0, z = last RHS. Check that the RHS are ≥ 0 and that z did not decrease. Go back to step 2.

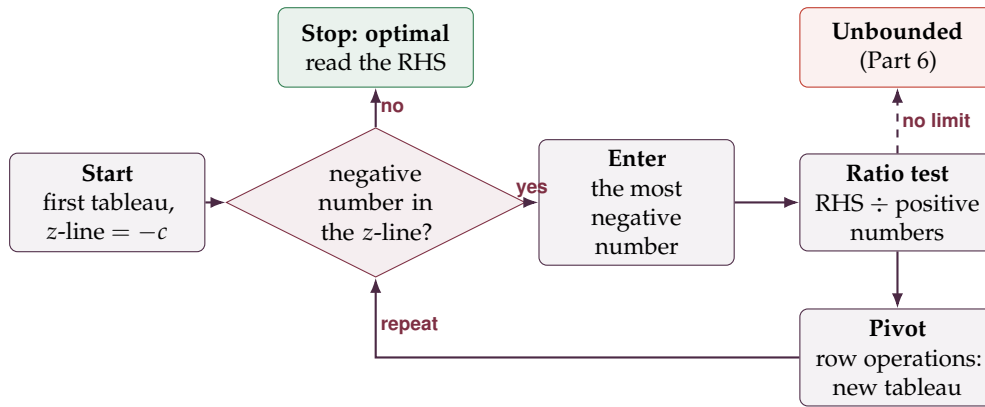


Figure 21: Method 3.2 as a picture: the same picture as Method 3.1, with tableau words.

Other books. Some books write the last line with the opposite signs (the line " $c_j - z_j$ "). Then the entering column is the *largest positive* number, and the tableau is optimal when all the numbers of this line are ≤ 0 . It is exactly the same method. Use fractions, not decimals: $\frac{1}{3}$ written as 0.33 quickly gives wrong answers.

Part 4 in short

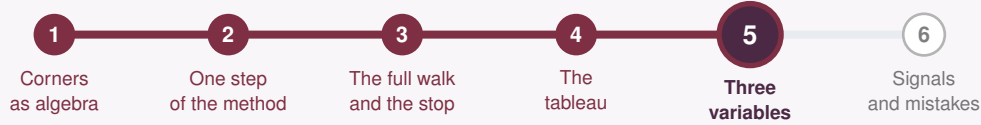
The tableau is the dictionary with all the variables on the left, so the signs of the z-line are opposite. Rule 3.2: the most negative number enters; the smallest ratio $\text{RHS} \div (\text{positive number})$ gives the pivot; row operations make the entering column a unit column. Two checks after each pivot: the RHS stay ≥ 0 , and z never decreases.

Check yourself

Q4. In pivot 2 of the drone workshop, compute the new line $L_2 \leftarrow L_2 - 3L_1$ yourself, and compare with tableau T2.

Part 5 of 6

Three variables: no picture needed



In this part, you learn to:

- ▶ run the tableau on a problem we cannot draw;
- ▶ read the plan and the unused resources;
- ▶ check the answer in the original constraints.

21 Example 3.2: three robot kits

The problem. A workshop builds three robot kits K1, K2, K3; x_j is the number of kits K_j made per day.

	K1	K2	K3	Available per day
Cutting (hours per kit)	3	2	2	24
Assembly (hours per kit)	2	2	2	30
Testing (hours per kit)	3	1	2	20
Profit (thousand DA per kit)	3	3	5	

$$\max z = 3x_1 + 3x_2 + 5x_3$$

$$\text{s.t. } 3x_1 + 2x_2 + 2x_3 \leq 24, \quad 2x_1 + 2x_2 + 2x_3 \leq 30, \quad 3x_1 + x_2 + 2x_3 \leq 20, \quad x_1, x_2, x_3 \geq 0.$$

With three variables, the feasible set is a shape in three dimensions: hard to draw, and with more variables impossible. The simplex method does not need any picture. All the right-hand sides are ≥ 0 , so we can start at the origin. We add the slacks s_1, s_2, s_3 (unused cutting, assembly and testing hours).

22 Example 3.2: the two pivots

Pivot 1 (Figure 22). The most negative number in the z -line is -5 , so x_3 enters (the kit with the largest profit). Ratios: $24/2 = 12$, $30/2 = 15$, $20/2 = 10$. The pivot is the 2 in L_3 ; s_3 leaves (testing is the first resource to run out). Row operations: first $L_3 \leftarrow L_3/2 = (\frac{3}{2}, \frac{1}{2}, 1, 0, 0, \frac{1}{2} \mid 10)$; then $L_1 \leftarrow L_1 - 2L_3$, $L_2 \leftarrow L_2 - 2L_3$, $z \leftarrow z + 5L_3$. For example,

$$L_1 = (3, 2, 2, 1, 0, 0 \mid 24) - 2 \times (\frac{3}{2}, \frac{1}{2}, 1, 0, 0, \frac{1}{2} \mid 10) = (0, 1, 0, 1, 0, -1 \mid 4).$$

In T1, $z = 50$; the z -line still has a negative number, $-\frac{1}{2}$ under x_2 .

		enters ↓							
leaves	T0	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio
	s_1	3	2	2	1	0	0	24	$24/2 = 12$
	s_2	2	2	2	0	1	0	30	$30/2 = 15$
	s_3	3	1	2	0	0	1	20	$20/2 = 10$
	z	-3	-3	-5	0	0	0	0	
pivot ↓									
		x_1	x_2	x_3	s_1	s_2	s_3	RHS	
	s_1	0	1	0	1	0	-1	4	$L_1 - 2L_3$
	s_2	-1	1	0	0	1	-1	10	$L_2 - 2L_3$
	x_3	$\frac{3}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	10	$L_3 \div 2$
	z	$\frac{9}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{5}{2}$	50	$z + 5L_3$

Figure 22: Example 3.2, pivot 1: x_3 enters, s_3 leaves.

Pivot 2 (Figure 23). The only negative number is $-\frac{1}{2}$: x_2 enters. Ratios: $4/1 = 4$, $10/1 = 10$, $10/\frac{1}{2} = 20$ (careful: dividing by $\frac{1}{2}$ doubles). The pivot is the 1 in L_1 ; s_1 leaves. Row operations: L_1 unchanged; $L_2 \leftarrow L_2 - L_1$; $L_3 \leftarrow L_3 - \frac{1}{2}L_1$; $z \leftarrow z + \frac{1}{2}L_1$. There is no negative number in the z -line of T2: it is optimal. (Check: z went from 0 to $0 + 5 \times 10 = 50$, then to $50 + \frac{1}{2} \times 4 = 52$.)

		enters $\downarrow$							
	T1	x_1	x_2	x_3	s_1	s_2	s_3	RHS	ratio
leaves	s_1	0	1	0	1	0	-1	4	$4/1 = 4$
	s_2	-1	1	0	0	1	-1	10	$10/1 = 10$
	x_3	$\frac{3}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	10	$10/\frac{1}{2} = 20$
	z	$\frac{9}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{5}{2}$	50	
pivot $\downarrow$									
	T2	x_1	x_2	x_3	s_1	s_2	s_3	RHS	
	x_2	0	1	0	1	0	-1	4	$L_1 \div 1$
	s_2	-1	0	0	-1	1	0	6	$L_2 - L_1$
	x_3	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	1	8	$L_3 - \frac{1}{2}L_1$
	z	$\frac{9}{2}$	0	0	$\frac{1}{2}$	0	2	52	$z + \frac{1}{2}L_1$

Figure 23: Example 3.2, pivot 2: x_2 enters, s_1 leaves. T2 is optimal.

23 Example 3.2: reading the answer

This is the most important part: an answer is not a tableau, it is a plan.

- ▶ Basic variables: $x_2 = 4, s_2 = 6, x_3 = 8$; non-basic: $x_1 = s_1 = s_3 = 0$.
- ▶ *Plan*: no kit K1, 4 kits K2 and 8 kits K3 per day, for a profit of $z = 52$ thousand DA.
- ▶ Cutting and testing are fully used ($s_1 = s_3 = 0$); 6 assembly hours are unused ($s_2 = 6$), see Figure 24.
- ▶ *Why no kit K1?* The z -line reads $z = 52 - \frac{9}{2}x_1 - \frac{1}{2}s_1 - 2s_3$: making one kit K1 would *lower* the profit by 4.5 thousand DA, because K1 uses many cutting and testing hours, which are worth more for K2 and K3.
- ▶ *Check* in the original constraints: cutting $0 + 8 + 16 = 24 \leq 24$; assembly $0 + 8 + 16 = 24 \leq 30$; testing $0 + 4 + 16 = 20 \leq 20$; profit $0 + 12 + 40 = 52$.

Here the optimal plan is made of whole numbers; as explained in Lecture 1 (Section 2.7), this is luck.



Figure 24: Use of the resources in the best plan of Example 3.2: cutting and testing are fully used, 6 of the 30 assembly hours are left.

Part 5 in short

With three variables nothing changes: the same three questions, the same row operations, no picture. The answer is a plan in words (how many kits of each type), the unused resources (the slacks), the profit, and a check in the original constraints. A positive number under a non-basic variable in the final z -line tells how much the profit would drop for one unit of it.

Check yourself

Q5. From tableau T2 of Example 3.2, explain in two lines why no feasible plan gives more than 52 thousand DA.

Q6. In pivot 1 of Example 3.2, compute the new line $L_2 \leftarrow L_2 - 2L_3$ yourself, and compare with tableau T1.

Part 6 of 6

Signals, why it works, and mistakes



In this part, you learn to:

- ▶ recognise an unbounded problem in the tableau;
- ▶ recognise many optimal plans in the tableau;
- ▶ explain why the method works, and avoid the common mistakes.

In Lecture 2 we met five cases. The simplex method recognises them with signals in the tableau. Two of them appear today: an unbounded problem, and many optimal plans. (An infeasible problem is recognised when we look for a first corner: Lecture 4.)

24 Signal 1: no ratio $\rightarrow$ the problem is unbounded

Consider $\max z = 2x_1 + x_2$ subject to $x_1 - x_2 \leq 2$, $-x_1 + x_2 \leq 1$, $x_1, x_2 \geq 0$. In the first tableau, x_1 enters (-2); the only ratio is $2/1 = 2$ in line 1 (line 2 has -1). After this pivot ($L_2 \leftarrow L_2 + L_1$, $z \leftarrow z + 2L_1$) we get the tableau T1 of Figure 25. Now x_2 should enter (-3), but its column has no positive number (-1 and 0): there is no ratio, so nothing stops x_2 .

Let us see what happens. With $x_2 = t \geq 0$ and $s_1 = 0$, the lines of the tableau give

$$x_1 - t = 2 \Rightarrow x_1 = 2 + t, \quad s_2 = 3, \quad z - 3t = 4 \Rightarrow z = 4 + 3t.$$

All these plans are feasible for every $t \geq 0$: check in the original constraints, $x_1 - x_2 = 2 \leq 2$ and $-x_1 + x_2 = -2 \leq 1$. And $z = 4 + 3t \rightarrow +\infty$: z is 4, then 7, 10, 13, ... So the problem is *unbounded* (Lecture 2, Case 4). In a real problem, this usually means that a constraint is missing in the model.

		enters ↓				
T1	x_1	x_2	s_1	s_2	RHS	ratio
x_1	1	-1	1	0	2	-1: no ratio
s_2	0	0	1	1	3	0: no ratio
z	0	-3	2	0	4	

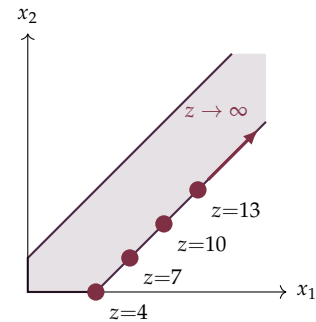


Figure 25: Signal 1. Left: in T1, x_2 should enter but its column has no positive number: no ratio. Right: along $x_1 = 2 + t$, $x_2 = t$, the plans stay feasible and $z = 4 + 3t$ grows without limit.

Rule 3.3 — Unbounded problem

If the entering column has no positive number (above the z -line), the problem is unbounded. The column gives a family of feasible plans along which $z \rightarrow +\infty$.

Why? If no number of the entering column is positive, then when the entering variable grows, no basic variable decreases. So no variable ever becomes negative: we can go as far as we want, and z grows all the time.

25 Signal 2: a 0 in the z -line $\rightarrow$ many optimal plans

Take the drone workshop with a profit of 30 thousand DA for a drone A: $\max z = 30x_1 + 60x_2$. The first two pivots are the same as before (x_2 enters, then x_1), and we reach the corner C with the tableau T2 of Figure 26. There is no negative number in its z -line: C = (4, 18) is optimal, with $z = 1200$.

But look at the non-basic variable s_3 : it has a 0 in the z -line. The z -line reads $z = 1200 - 30s_1 - 0 \times s_3$, so if s_3 enters, z does not change. Doing this pivot (ratios 3 and 18, so s_2 leaves) gives the corner B = (10, 15), also with $z = 30 \times 10 + 60 \times 15 = 1200$. So C and B are both optimal, and all the points of the edge between them are optimal too (Lecture 2, Case 2). On the picture, the profit lines $30x_1 + 60x_2 = \text{constant}$ are parallel to the assembly line, and the best one lies exactly on the edge C-B.

0: try it ↓

T2	x_1	x_2	s_1	s_2	s_3	RHS	ratio
x_1	1	0	1	0	-2	4	-2: no ratio
leaves s_2	0	0	-3	1	4	12	$12/4 = 3$
x_2	0	1	0	0	1	18	$18/1 = 18$
z	0	0	30	0	0	1200	

pivot (z stays 1200)

T3	x_1	x_2	s_1	s_2	s_3	RHS	
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	10	$L_1 + 2L_2$
s_3	0	0	$-\frac{3}{4}$	$\frac{1}{4}$	1	3	$L_2 \div 4$
x_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	15	$L_3 - L_2$
z	0	0	30	0	0	1200	unchanged

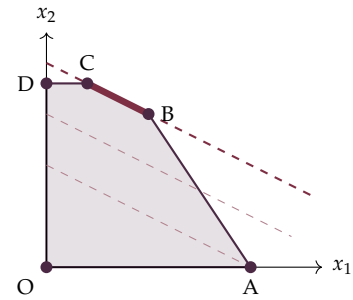


Figure 26: Signal 2. Left: T2 is optimal, but there is a 0 under the non-basic s_3 ; the pivot on this column gives T3, another optimal corner with the same $z = 1200$. Right: the profit lines (dashed) are parallel to the edge C–B, and the best one contains the whole edge.

Rule 3.4 — Many optimal plans

If an optimal tableau has a 0 in the z -line under a *non-basic* variable, then in general other optimal plans exist: a pivot on this column gives another optimal corner, and all the points between the two corners are optimal.

(“In general”: if the ratio of this pivot is 0, the pivot changes the basic variables but not the point; this is the degenerate situation of Lecture 4.)

26 Why the simplex method works

We can now explain why the method always gives the right answer.

1. **Each tableau is a corner.** The ratio test keeps all the $\text{RHS} \geq 0$ (Section 20), so each tableau gives a basic *feasible* solution, that is a corner (Key fact 3.1).
2. **z never decreases.** At each pivot, z grows by (smallest ratio) $\times$ (size of the entering number): see Figure 27.
3. **It stops.** If z grows strictly at each pivot, we never come back to a corner visited before (it would have a smaller value), and there are only finitely many corners (at most $\binom{n+m}{m}$ basic solutions with n variables and m constraints). So after a finite number of pivots, we stop.
4. **The stop is a proof.** We stop in two situations only: either no number of the z -line is negative, and then the last z -line says that z equals its final value z^* minus a sum of terms (non-negative coefficient) $\times$ (non-negative variable), so $z \leq z^*$ for every feasible plan (Key fact 3.2); or the entering column has no positive number, and the problem is unbounded (Rule 3.3).

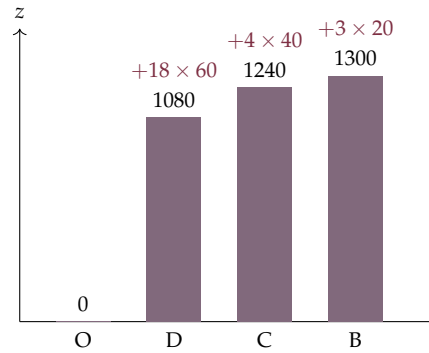


Figure 27: The value of z along the walk of the drone workshop: each pivot adds (ratio) $\times$ (entering number).

One exception. If the smallest ratio is 0, z does not grow at this pivot. This happens at a *degenerate* corner, where more constraint lines than necessary pass through the corner (Lecture 2, Section 8.2). Then the method could, in rare cases, turn in a circle. Degeneracy, and how to start when the origin is not feasible, are studied in Lecture 4.

In practice the simplex method is very fast: it usually visits only a small number of corners, even for problems with thousands of variables.

27 Common mistakes

Wrong ✗	Right ✓
A ratio with a 0 or a negative number in the entering column	Only positive numbers (in the tableau) limit the entering variable
The largest ratio, or the line with the largest number in the column	Always the smallest ratio
Divide only part of the pivot line	Divide the whole pivot line, RHS included
Forget to update the z-line	The z-line is a line like the others
Forget to change the name of the basic variable of the pivot line	The entering variable becomes the basic variable of the pivot line
A negative number appears in the RHS column	Impossible (the RHS are the values of the basic variables): there is a mistake in the pivot or in the arithmetic
Read the sign of the z-line the wrong way	Tableau: negative means “can improve”; dictionary: positive means “can improve”
Give only numbers	Give the plan in words, the unused resources and z , and check the plan in the original constraints

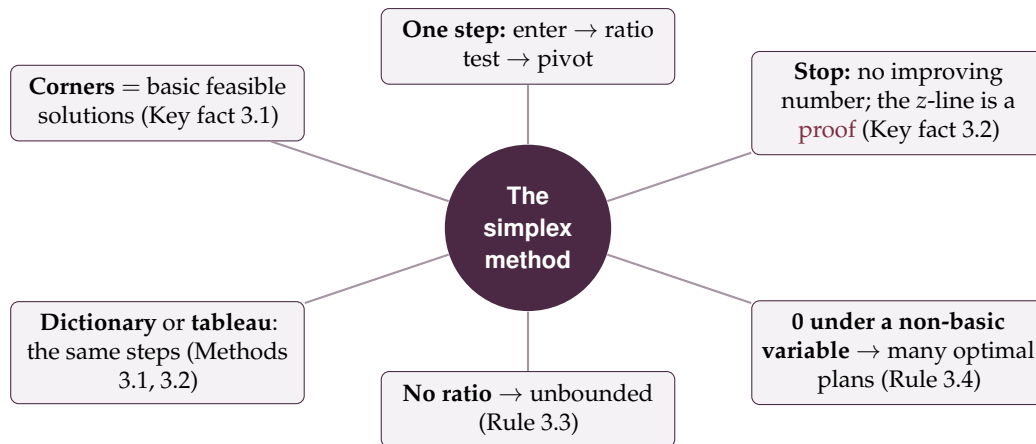
Part 6 in short

No positive number in the entering column $\rightarrow$ unbounded (Rule 3.3). A 0 under a non-basic variable in the optimal z-line $\rightarrow$ in general, many optimal plans (Rule 3.4). The method works because each tableau is a corner, z never decreases, there are finitely many corners, and the final z-line is a proof. The one exception, a ratio 0 (degeneracy), is for Lecture 4.

Check yourself

Q7. In the unbounded example, give the feasible plan for $t = 10$ and its value of z .

Summary: the simplex method in one picture



- ▶ A basic solution: m basic variables, the others equal to 0 (Definition 3.1). The basic feasible solutions are the corners (Key fact 3.1). Neighbouring corners differ by one exchange: one variable enters, one leaves.
- ▶ A dictionary writes the basic variables and z with the non-basic variables (Definition 3.2).
- ▶ One step = three questions: who enters (Rule 3.1), how far (smallest ratio, the first wall on our way), and the pivot (substitution or row operations).
- ▶ Stop when no coefficient can improve z : the last z -line is a proof of optimality (Key fact 3.2), and its coefficients are the multipliers of Lecture 2.
- ▶ Dictionary (Method 3.1) and tableau (Method 3.2, Rule 3.2): the same steps, written differently. Checks: $\text{RHS} \geq 0$, z never decreases.
- ▶ Signals: no positive number in the entering column $\rightarrow$ unbounded (Rule 3.3); a 0 under a non-basic variable in the optimal z -line $\rightarrow$ many optimal plans (Rule 3.4).

What comes next. Today the origin was always a corner to start from. In Lecture 4 we learn how to start when the origin is not feasible (constraints “ $\geq$ ” and “ $=$ ”), how to recognise an infeasible problem, and what to do at a degenerate corner.

Reading. F. S. Hillier and G. J. Lieberman, *Introduction to Operations Research*, Chapter 4 (Sections 4.1–4.4); H. A. Taha, *Operations Research: An Introduction*, Chapter 3.

Answers to “Check yourself”

Q1 (Part 1). With $x_2 = s_2 = 0$: the second equation gives $3x_1 = 60$, so $x_1 = 20$; then $s_1 = 40 - 20 = 20$ and $s_3 = 18$. The basic solution $(20, 0, 20, 0, 18)$ is feasible: it is the corner A.

Q2 (Part 2). With $x_2 = 0$ and $x_1 = t$: $s_1 = 40 - t$ gives $t \leq 40$; $s_2 = 60 - 3t$ gives $t \leq 20$; $s_3 = 18$ gives no limit. The smallest ratio is 20: s_2 leaves, and we reach the corner A $= (20, 0)$ with $z = 800$. (Continuing, x_2 enters and we reach B in one more pivot: TD 3.)

Q3 (Part 3). The last dictionary gives $x_1 = 10 + \frac{1}{2} \times 1 - 0 = 10.5$, $x_2 = 15 - \frac{3}{4} \times 1 + 0 = 14.25$, and $z = 1300 - 25 \times 1 - 0 = 1275$. Check: $40 \times 10.5 + 60 \times 14.25 = 420 + 855 = 1275$. One unused

assembly hour costs 25 thousand DA, as the z-line says. (Also $s_3 = 3 + \frac{3}{4} = 3.75$; this is the LP relaxation, so fractional values are allowed here.)

Q4 (Part 4). $L_2 - 3L_1 = (3, 0, 0, 1, -2 \mid 24) - 3 \times (1, 0, 1, 0, -2 \mid 4) = (0, 0, -3, 1, 4 \mid 12)$: the line s_2 of T2.

Q5 (Part 5). The z-line of T2 reads $z = 52 - \frac{9}{2}x_1 - \frac{1}{2}s_1 - 2s_3$. For every feasible plan, $x_1, s_1, s_3 \geq 0$, so $z \leq 52$; and the plan $(0, 4, 8)$ gives 52.

Q6 (Part 5). $L_2 - 2L_3 = (2, 2, 2, 0, 1, 0 \mid 30) - 2 \times (\frac{3}{2}, \frac{1}{2}, 1, 0, 0, \frac{1}{2} \mid 10) = (-1, 1, 0, 0, 1, -1 \mid 10)$: the line s_2 of T1.

Q7 (Part 6). For $t = 10$: $x_1 = 12, x_2 = 10, z = 2 \times 12 + 10 = 34 = 4 + 3 \times 10$. Check: $12 - 10 = 2 \leq 2$ and $-12 + 10 = -2 \leq 1$.