

Lecture 2 — Solving a Linear Program

The graphical method, the corner method, and all the possible cases



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Operations Research (O.R.)

What you will learn. In Lecture 1 we learned to *write* linear programs. Today we learn to *solve* them when there are two variables. You will be able to:

- ▶ draw the feasible set of a linear program with two variables;
- ▶ find the best plan by drawing (the graphical method) and by computing the corners (the corner method);
- ▶ prove that a plan is the best, with a short computation;
- ▶ recognise every case that can happen: one best plan, many best plans, no best plan, no feasible plan, and three special situations.

With two variables we can see everything on a picture. This is the best way to understand what happens in larger problems, which we solve with the simplex method in Lecture 3.

How to read this lecture. Each idea starts with an example with numbers, then the general rule, then the reason why it is true. Methods and key facts are in boxes. The “Check yourself” questions have their answers at the end.

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1 A plan is a point of the plane

We use the drone workshop of Lecture 1:

$$\begin{aligned}
 \max \quad & z = 40x_1 + 60x_2 \quad (\text{profit, thousand DA per week}) \\
 \text{s.t.} \quad & x_1 + 2x_2 \leq 40 \quad (\text{assembly hours}) \\
 & 3x_1 + 2x_2 \leq 60 \quad (\text{testing hours}) \\
 & x_2 \leq 18 \quad (\text{cameras}) \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

With two variables, a plan (x_1, x_2) is a *point* of the plane: x_1 is read on the horizontal axis and x_2 on the vertical axis. Each constraint is satisfied by the points on *one side of a line*, and the feasible set (Lecture 1, Definition 1.2) is the region where *all* the constraints are satisfied. So we can draw the feasible set, and then look for its best point.

To do this, we need three simple tools: draw a line, choose the correct side of a line, and compute the point where two lines meet.

2 Three tools

2.1 Tool 1: draw a line

The constraint $3x_1 + 2x_2 \leq 60$ is limited by the line $3x_1 + 2x_2 = 60$. A line is fixed by two of its points. The easiest points are on the axes:

- ▶ put $x_1 = 0$: then $2x_2 = 60$, so $x_2 = 30$; this gives the point $(0, 30)$;
- ▶ put $x_2 = 0$: then $3x_1 = 60$, so $x_1 = 20$; this gives the point $(20, 0)$.

Join the two points with a ruler. Three kinds of lines need attention:

- ▶ $x_2 = 18$ is a *horizontal* line: all the points whose second coordinate is 18;
- ▶ $x_1 = 5$ is a *vertical* line: all the points whose first coordinate is 5;
- ▶ $x_1 - x_2 = 0$ passes through the origin: both points on the axes are $(0, 0)$, so we need another point, for example $(1, 1)$.

2.2 Tool 2: choose the correct side

A line cuts the plane into two half-planes. A constraint " $\leq$ " or " $\geq$ " keeps one of them. There are two ways to find which one.

First way: the test point. Take a point that is *not* on the line, and check the constraint at this point:

- ▶ if the constraint is *true* at the test point, keep the side of the line that contains the test point;
- ▶ if it is *false*, keep the other side.

The origin $(0, 0)$ is usually the easiest test point, because the computation gives 0.

- ▶ $3x_1 + 2x_2 \leq 60$ at $(0, 0)$: $0 \leq 60$ is true. We keep the side of the origin.
- ▶ $x_1 + x_2 \geq 6$ at $(0, 0)$: $0 \geq 6$ is false. We keep the other side, away from the origin.
- ▶ $x_2 \leq 2x_1$: the origin is *on* the line, so it cannot be a test point. We use $(1, 0)$: $0 \leq 2$ is true, so we keep the side of $(1, 0)$.

Why does it work? On one side of the line the expression is smaller than the right-hand side, on the other side it is larger, and it can only change from one to the other by crossing the line. So the answer at one test point is the answer for the whole side.

Second way: the arrow of coefficients. For the constraint $3x_1 + 2x_2 \leq 60$, take its coefficients 3 and 2 and draw the arrow $a = (3, 2)$ from a point of the line (Figure 1). This arrow has two properties:

- ▶ *In the direction of the arrow, the expression $3x_1 + 2x_2$ grows.* If we move a point (p_1, p_2) by the arrow, to $(p_1 + 3, p_2 + 2)$, the expression becomes $3(p_1 + 3) + 2(p_2 + 2) = (3p_1 + 2p_2) + 13$: it grows by $13 = 3^2 + 2^2$.
- ▶ *The arrow is perpendicular to the line.* If we move a point by $(-2, 3)$, the expression changes by $3 \times (-2) + 2 \times 3 = 0$: so $(-2, 3)$ is the direction of the line. And $(3, 2)$ is perpendicular to $(-2, 3)$, because $3 \times (-2) + 2 \times 3 = 0$.

Consequence: a constraint " $\leq$ " keeps the side *opposite* to the arrow, and a constraint " $\geq$ " keeps the side *of* the arrow. The same arrow will tell us later in which direction the profit grows.

Common mistake

For a " $\geq$ " constraint, many students keep the side of the origin by habit. Always do the test: for $x_1 + x_2 \geq 6$, the origin gives $0 \geq 6$, which is false, so the origin is *not* on the correct side.

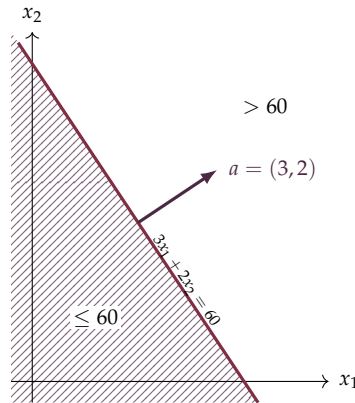


Figure 1: The arrow of coefficients $a = (3, 2)$ is perpendicular to the line and points to where $3x_1 + 2x_2$ grows. The constraint $3x_1 + 2x_2 \leq 60$ keeps the hatched side.

2.3 Tool 3: compute a corner

The corners of the feasible set are points where two lines meet. We never read their coordinates on the drawing (a drawing is never exact): we compute them.

Method 2.1 — Compute a corner

A corner is a point where two lines meet. Write the two equations and solve the system (by elimination or by substitution). Always check the result in both equations.

By elimination. Solve $x_1 + 2x_2 = 40$ (1) and $3x_1 + 2x_2 = 60$ (2). Both equations contain $2x_2$, so we subtract them: (2) – (1) gives $2x_1 = 20$, so $x_1 = 10$. Then (1) gives $10 + 2x_2 = 40$, so $x_2 = 15$. Check in (2): $30 + 30 = 60$. The two lines meet at (10, 15).

By substitution. Solve $2x_1 + x_2 = 8$ and $x_1 + 2x_2 = 7$. From the first equation, $x_2 = 8 - 2x_1$. We put it in the second: $x_1 + 2(8 - 2x_1) = 7$, so $x_1 + 16 - 4x_1 = 7$, then $-3x_1 = -9$ and $x_1 = 3$; finally $x_2 = 8 - 6 = 2$. Check: $6 + 2 = 8$ and $3 + 4 = 7$. The lines meet at (3, 2).

Check yourself

Q1. Draw the line $x_1 + 3x_2 = 12$: give its two points on the axes. Which side does $x_1 + 3x_2 \leq 12$ keep?

Q2. Compute the point where the lines $2x_1 + x_2 = 10$ and $x_1 - x_2 = 2$ meet.

3 The graphical method on the drone workshop

3.1 Step 1: draw the feasible set

We draw the three lines:

- ▶ assembly $x_1 + 2x_2 = 40$, through (40, 0) and (0, 20);
- ▶ testing $3x_1 + 2x_2 = 60$, through (20, 0) and (0, 30);
- ▶ cameras $x_2 = 18$, a horizontal line.

The origin satisfies the three constraints ($0 \leq 40$, $0 \leq 60$, $0 \leq 18$), so each time we keep the side of the origin. The sign constraints $x_1, x_2 \geq 0$ keep the part of the plane to the right of the vertical axis

and above the horizontal axis. Figure 2 shows how the feasible set is built, one constraint at a time: each constraint keeps a half-plane, and the feasible set is the part common to all of them.

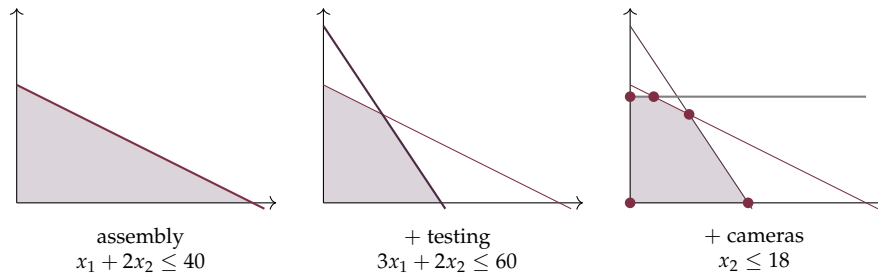


Figure 2: Building the feasible set of the drone workshop, one constraint at a time (with $x_1, x_2 \geq 0$).

The feasible set is the polygon O A B C D with five corners. We compute each corner with Method 2.1:

- ▶ O = (0,0): the two axes;
- ▶ A = (20,0): the testing line and the axis $x_2 = 0$ ($3x_1 = 60$);
- ▶ B = (10,15): the assembly and testing lines (computed above);
- ▶ C = (4,18): the assembly and camera lines ($x_2 = 18$, then $x_1 + 36 = 40$);
- ▶ D = (0,18): the camera line and the axis $x_1 = 0$.

3.2 Step 2: lines of equal profit

All the plans with the same profit k are on the line $40x_1 + 60x_2 = k$. For example:

- ▶ the plans with profit $k = 600$ are on the line through (15,0) and (0,10);
- ▶ the plans with profit $k = 900$ are on the line through (22.5,0) and (0,15).

These lines are all *parallel*: they have the same coefficients 40 and 60, only the number k changes. When k grows, the line moves in the direction of the arrow $c = (40, 60)$ made of the coefficients of z (this is Tool 2, second way, applied to the objective).

3.3 Step 3: push the line as far as possible

We slide the line of equal profit parallel to itself, in the direction of c , as far as possible *while it still touches the feasible set* (Figure 3). The last point touched is the corner B = (10,15), on the line $z = 1300$.

Why is B the best plan? A line with a profit $k > 1300$ is further in the direction of c , and it does not touch the feasible set: no feasible plan has a profit $k > 1300$. And B is feasible with profit $40 \times 10 + 60 \times 15 = 1300$.

Answer. The optimal plan is 10 drones A and 15 drones B per week; the optimal value is 1300 thousand DA. These are whole numbers: as explained in Lecture 1 (Section 2.7), this is luck, and here the best plan of the LP relaxation is also the best plan of the real problem.

Common mistake

Reading the best point on the drawing is not enough: a drawing is never exact. The drawing tells you *which* lines meet at the best corner; then you *compute* the corner with Method 2.1.

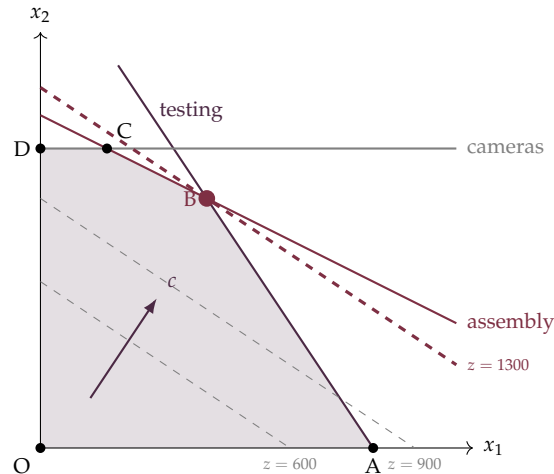


Figure 3: The drone workshop: feasible set O A B C D, lines of equal profit, and the optimal corner B.

3.4 Reading the solution

A good answer does not stop at the numbers: it explains what the plan uses.

Constraint	Used at B	Available	Unused (slack)	
Assembly	$10 + 30 = 40$	40	0	binding
Testing	$30 + 30 = 60$	60	0	binding
Cameras	15	18	3	not binding

A constraint is *binding* (or *active*) when it is satisfied with equality at the optimal plan: its line passes through the optimal point, and its slack is 0. Binding constraints are the bottlenecks of the problem: to earn more, the workshop needs more assembly hours or more testing hours. More cameras would not help, since 3 cameras are already unused.

4 Why the best plan is at a corner

In the drone workshop the best plan is a corner. This is not a coincidence:

Key fact 2.1

If a linear program has an optimal solution, then at least one corner of the feasible set is an optimal solution.

This fact is the basis of all the methods of this course: to find the best plan, we only need to look at the corners, and there are finitely many of them. We explain it in three steps.

4.1 The feasible set has no holes and no dents

If two plans P and Q are feasible, then every plan on the segment between them is also feasible. The points of the segment are $M = (1 - t)P + tQ$ with t between 0 and 1 ($t = 0$ gives P , $t = 1$ gives Q , $t = \frac{1}{2}$ gives the middle point). Take one constraint $a_1x_1 + a_2x_2 \leq b$. Its left side at M is

$$(1 - t) \times (\text{its value at } P) + t \times (\text{its value at } Q) \leq (1 - t)b + tb = b,$$

because both values are at most b and the numbers $1 - t$ and t are not negative. So M satisfies the constraint. This is true for every constraint, so M is feasible. (Lecture 1, Key fact 1.1, showed the same thing for the middle point.)

A set with this property is called *convex* (Figure 4). *The feasible set of a linear program is always convex.*

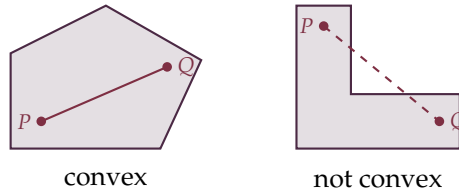


Figure 4: Left: a convex set, the segment between two of its points stays inside. Right: a set that is not convex. The feasible set of a linear program always looks like the left picture.

4.2 On a segment, the value changes regularly

Take the edge from $C = (4, 18)$ to $B = (10, 15)$. To go from C to B we add 6 to x_1 and -3 to x_2 , so the points of the edge are $(4 + 6t, 18 - 3t)$, with t from 0 (point C) to 1 (point B). Their profit is

$$40(4 + 6t) + 60(18 - 3t) = 160 + 240t + 1080 - 180t = 1240 + 60t.$$

This is a straight-line function of t (of the form $\alpha + \beta t$). When t goes from 0 to 1, it goes regularly from 1240 to 1300. So on this edge, the best value is at one end, here at B .

The same computation works on *any* segment, for *any* linear objective: on the segment from P to Q , the value is $z(P) + t(z(Q) - z(P))$. It increases, decreases, or stays constant; in every case, *the best value on a segment is reached at one of its two ends*.

4.3 From any plan to a corner

Here is the idea on the drone workshop (Figure 5).

1. Start from any feasible plan, for example $(6, 6)$, with $z = 600$.
2. Move in a direction where the profit grows. We take the direction $(2, 3)$, which is the direction of $c = (40, 60)$. The points are $(6 + 2t, 6 + 3t)$. How far can we go? Testing gives $3(6 + 2t) + 2(6 + 3t) = 30 + 12t \leq 60$, so $t \leq 2.5$; assembly would allow $t \leq 2.75$ and cameras $t \leq 4$. The first constraint to stop us is testing: we stop at $t = 2.5$, at the point $(11, 13.5)$ on the testing line, with $z = 440 + 810 = 1250$.
3. Now we are on an edge. On this edge the value changes regularly (previous subsection), so in one of the two directions it does not decrease. Here the direction $(-2, 3)$ stays on the testing line ($3 \times (-2) + 2 \times 3 = 0$) and changes z by $40 \times (-2) + 60 \times 3 = 100 > 0$ per step. We follow it until the end of the edge: the corner $B = (10, 15)$, with $z = 1300$.

So from the plan $(6, 6)$ we reached a corner which is at least as good. The same is true from any feasible plan: move in the direction of c until a border, then along the border until a corner, never losing value. Therefore no feasible plan is better than the best corner, and this proves Key fact 2.1 for a polygon. (When the feasible set goes to infinity, an extra difficulty appears: there may be no best plan at all. This is Case 4 in Section 7.)

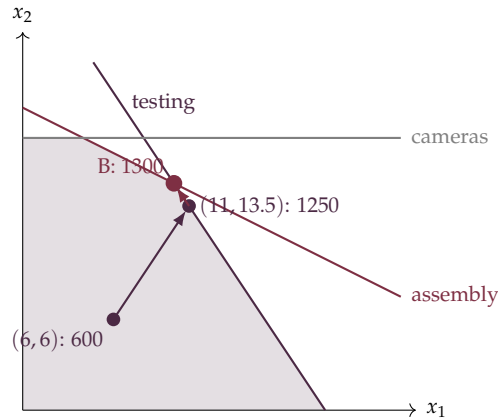


Figure 5: Sliding to a corner: from $(6,6)$ in the direction of c to the border, then along the border to the corner B.

4.4 The corner method

Key fact 2.1 gives a second method, which uses only computations: compute the value at every corner and take the best one.

Method 2.2 — The corner method

1. Draw the feasible set and compute all its corners (Method 2.1).
2. Compute z at each corner.
3. Choose the best corner.

Warning: if the feasible set goes to infinity, first check that an optimal solution exists (see Cases 3 and 4).

For the drone workshop:

Corner	(x_1, x_2)	$z = 40x_1 + 60x_2$
O	$(0,0)$	0
A	$(20,0)$	800
B	$(10,15)$	1300
C	$(4,18)$	1240
D	$(0,18)$	1080

The best corner is B, with 1300 thousand DA, as with the graphical method.

5 How can we be sure that a plan is optimal?

A picture is convincing, but a picture can be wrong. Here is a way to be *completely sure* that no plan is better than B, with simple algebra. It uses two facts about inequalities:

- ▶ an inequality keeps its direction when we multiply both sides by a *positive* number;
- ▶ two inequalities in the same direction can be added.

So if we multiply some constraints by positive numbers and add them, we get a new inequality, which *every feasible plan satisfies*.

Try 1. Multiply the assembly constraint by 40: $40x_1 + 80x_2 \leq 1600$. Since $x_2 \geq 0$, we have $60x_2 \leq 80x_2$, so for every feasible plan

$$40x_1 + 60x_2 \leq 40x_1 + 80x_2 \leq 1600.$$

No plan gives more than 1600 thousand DA. This is true, but it does not prove that B is optimal: no plan reaches 1600.

Try 2. Take 25 times the assembly constraint plus 5 times the testing constraint:

$$25(x_1 + 2x_2) + 5(3x_1 + 2x_2) \leq 25 \times 40 + 5 \times 60, \quad \text{that is} \quad 40x_1 + 60x_2 \leq 1300.$$

No plan gives more than 1300 thousand DA, and the plan B = (10, 15) gives exactly 1300 thousand DA. So B is optimal. This is a complete proof, which anybody can check without a drawing.

How to find the numbers 25 and 5? Call them y_1 (for assembly) and y_2 (for testing). We want the left side $y_1(x_1 + 2x_2) + y_2(3x_1 + 2x_2)$ to be exactly the profit $40x_1 + 60x_2$. Comparing the coefficients of x_1 and of x_2 :

$$y_1 + 3y_2 = 40, \quad 2y_1 + 2y_2 = 60.$$

The second equation gives $y_1 = 30 - y_2$; then $30 - y_2 + 3y_2 = 40$, so $y_2 = 5$ and $y_1 = 25$. Both are positive, as needed. We used the two constraints that are binding at B: this is always the right choice, because only binding constraints can give a bound that B reaches.

What do these numbers mean? In Chapter 3 we will see that $y_1 = 25$ is the value of one more assembly hour (25 thousand DA), and $y_2 = 5$ the value of one more testing hour. They also come back in Lecture 3: the simplex method finds them automatically.

Common mistake

The multipliers must be *positive* (or zero). Multiplying a " $\leq$ " constraint by a negative number reverses it, and the sum no longer gives an upper bound.

6 The graphical method in general, and a minimisation

Method 2.3 — The graphical method

1. Draw each constraint line and keep the correct side (Tools 1 and 2).
2. Shade the feasible set. If it is empty, the problem is *infeasible*: stop.
3. Draw one line of equal value $z = k$ and the arrow c of the coefficients of z .
4. Push the line in the direction of c for a max, or in the *opposite* direction for a min, as far as possible while it still touches the feasible set.
5. If it can be pushed forever, the problem is *unbounded*: stop.
6. Otherwise read the last corner (or edge) touched, compute it exactly with Method 2.1, and give the optimal value z^* .

Example: spare-part boxes. A maintenance team buys boxes of spare parts for its robots. A box A contains 2 motors and 1 sensor and costs 30 thousand DA; a box B contains 1 motor and 2 sensors and costs 40 thousand DA. The team needs at least 8 motors and 7 sensors. With x_1, x_2 the numbers of boxes A and B:

$$\min z = 30x_1 + 40x_2 \quad \text{s.t.} \quad 2x_1 + x_2 \geq 8 \text{ (motors)}, \quad x_1 + 2x_2 \geq 7 \text{ (sensors)}, \quad x_1, x_2 \geq 0.$$

► *Lines.* $2x_1 + x_2 = 8$ through (4, 0) and (0, 8); $x_1 + 2x_2 = 7$ through (7, 0) and (0, 3.5).

- *Sides.* At $(0,0)$: $0 \geq 8$ is false and $0 \geq 7$ is false. So for both constraints we keep the side *away* from the origin. The feasible set goes to infinity, up and to the right (Figure 6).
- *Corners.* $(0,8)$ (motors line and vertical axis), $(7,0)$ (sensors line and horizontal axis), and the point where the two lines meet, $(3,2)$, computed in Section 2.3.
- *Push.* This is a minimisation, so we push the line $30x_1 + 40x_2 = k$ *towards the origin*, opposite to the arrow $c = (30,40)$. The last point touched is $(3,2)$.
- *Check with the corners.* $(0,8)$: 320; $(3,2)$: $90 + 80 = 170$; $(7,0)$: 210.

Answer. Buy 3 boxes A and 2 boxes B, for 170 thousand DA. Both constraints are binding: the team gets exactly 8 motors and 7 sensors.

Note that the feasible set goes to infinity, but the problem has a best plan: the costs are positive, so going far away only costs more. We come back to this in Case 3.

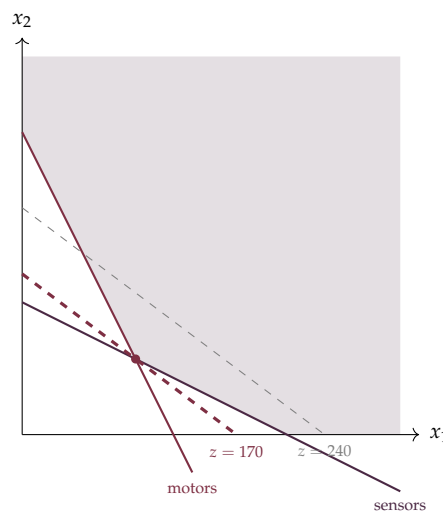


Figure 6: Spare-part boxes: the feasible set goes to infinity, and the cheapest plan is the corner $(3,2)$.

Check yourself

Q3. Check that $\frac{20}{3} \times (2x_1 + x_2 \geq 8) + \frac{50}{3} \times (x_1 + 2x_2 \geq 7)$ gives $30x_1 + 40x_2 \geq 170$ (combining constraints, as in Section 5). What does it prove?

7 All the possible cases

When we solve a linear program, exactly one of the following five cases happens.

Case	What we see	Answer
1. Unique optimum	the line touches one corner last	one optimal plan
2. Many optima	the line touches a whole edge last	infinitely many optimal plans
3. Unbounded set, optimum exists	the set goes to infinity, but the line stops	an optimal plan exists
4. Unbounded problem	the line can be pushed forever	no optimal plan
5. Infeasible problem	the feasible set is empty	no feasible plan

For each case we say how to recognise it and how to write the answer.

7.1 Case 1: a unique optimal solution

The line of equal value touches only one corner last. *Example:* the drone workshop, with the optimum $B = (10, 15)$. *Answer:* the plan and the value z^* .

7.2 Case 2: infinitely many optimal solutions

Suppose the profit of a drone A is 30 thousand DA instead of 40: $z = 30x_1 + 60x_2$. Since $30x_1 + 60x_2 = 30(x_1 + 2x_2)$, the lines of equal value are *parallel to the assembly line* $x_1 + 2x_2 = 40$. When we push them, the last contact is not one point, but the whole edge from C to B (Figure 7).

With the corner method: B gives $300 + 900 = 1200$ and C gives $120 + 1080 = 1200$, the same best value. By the computation of Section 4.2, all the points of the edge between C and B have the same value, so they are all optimal:

$$(4 + 6t, 18 - 3t), \quad 0 \leq t \leq 1, \quad z^* = 1200.$$

For example $t = \frac{1}{2}$ gives $(7, 16.5)$ with $z = 210 + 990 = 1200$.

How to recognise it: two neighbouring corners give the same best value, or the objective is parallel to a binding constraint. *How to write the answer:* give the two optimal corners and the formula of all the points between them. In practice this is good news: we can choose among many best plans (for example a plan with whole numbers, or the plan that uses fewer cameras).

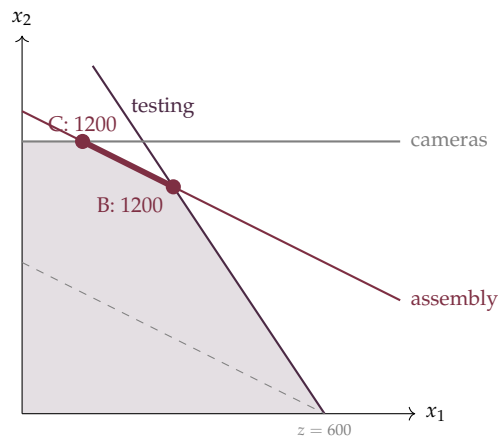


Figure 7: Case 2: with $z = 30x_1 + 60x_2$, the lines of equal value are parallel to the assembly line, and the whole edge C–B is optimal.

7.3 Cases 3 and 4: an unbounded feasible set

Take the feasible set $x_1 - x_2 \leq 1, x_1, x_2 \geq 0$ (Figure 8). It goes to infinity. Whether there is a best plan now depends on the objective.

Case 3: $\min x_1 + x_2$. For every feasible plan, $x_1 \geq 0$ and $x_2 \geq 0$, so $x_1 + x_2 \geq 0$; and $(0, 0)$ is feasible with value 0. So $(0, 0)$ is optimal, with $z^* = 0$. *An unbounded feasible set can have an optimal solution:* the spare-part boxes are another example.

Case 4: $\max x_1 + x_2$. Now the line can be pushed forever. To *prove* it, we give a family of feasible points whose value goes to infinity. For $t \geq 0$, the point $(1 + t, t)$ satisfies $(1 + t) - t = 1 \leq 1$, $1 + t \geq 0$ and $t \geq 0$: it is feasible. Its value is $1 + 2t$, which becomes as large as we want when t grows. So no plan can be optimal: whatever plan we propose, a point of the family is better. The problem is *unbounded*.

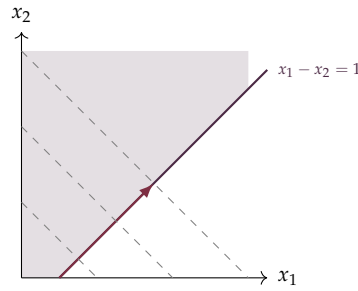


Figure 8: The feasible set $x_1 - x_2 \leq 1, x_1, x_2 \geq 0$ goes to infinity (Cases 3 and 4). The arrow shows the family of points $(1 + t, t)$; the dashed lines are lines $x_1 + x_2 = k$.

How to write the answer in Case 4: give a family of feasible points $x(t), t \geq 0$, check that they are feasible, and show that $z(x(t)) \rightarrow +\infty$ (or $-\infty$ for a minimisation). In practice, an unbounded problem almost always means that a constraint is missing in the model: no real workshop can earn an infinite profit.

Common mistake

“The feasible set goes to infinity, so the problem is unbounded” is *wrong*: Case 3 and the spare-part boxes have unbounded sets and a best plan. A problem is unbounded only when the *value* can go to infinity in the good direction.

7.4 Case 5: no feasible solution

$\max x_1 + x_2$ s.t. $x_1 + x_2 \leq 1, x_1 + x_2 \geq 2, x_1, x_2 \geq 0$. A feasible plan would need $2 \leq x_1 + x_2 \leq 1$, which is impossible. On the picture, the two half-planes do not meet (Figure 9). The feasible set is empty: the problem is *infeasible*.

How to write the answer: a short argument showing that no point satisfies all the constraints (often, as here, by combining two constraints). In practice: an error in the data or in the model, or a demand that is too high for the resources.

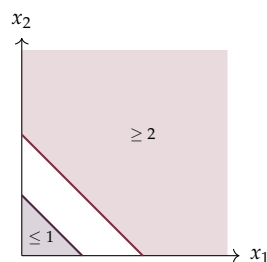


Figure 9: Case 5: the half-planes $x_1 + x_2 \leq 1$ and $x_1 + x_2 \geq 2$ do not meet; the feasible set is empty.

Check yourself

Q4. In the drone workshop, the profit of a drone B becomes 80 thousand DA (and A stays at 40 thousand DA). Which case is it? Give all the optimal plans.

Q5. Classify: $\max x_1 + 2x_2$ subject to $x_1 + x_2 \geq 4, x_1, x_2 \geq 0$.

8 Special situations

Three situations do not change the methods, but you must be able to recognise them and name them.

8.1 A redundant constraint

Add to the drone workshop the constraint “at most 25 drones A per week”: $x_1 \leq 25$. The testing constraint already gives $3x_1 \leq 60 - 2x_2 \leq 60$, so $x_1 \leq 20$ for every feasible plan. The line $x_1 = 25$ is outside the polygon: the new constraint removes no feasible plan, and the optimum does not change. Such a constraint is *redundant*: it can be removed without changing the problem. On the picture, a redundant constraint has its line outside the feasible set, or touching it at only one corner.

8.2 A degenerate corner

max $2x_1 + x_2$ subject to $x_1 \leq 4$, $x_2 \leq 2$, $x_1 + x_2 \leq 6$, $x_1, x_2 \geq 0$. The feasible set is the rectangle with corners $(0,0)$, $(4,0)$, $(4,2)$, $(0,2)$ (Figure 10). At $(4,2)$, *three* constraints are binding: $x_1 = 4$, $x_2 = 2$ and $x_1 + x_2 = 6$. In the plane, a corner where three or more constraint lines meet is called *degenerate*. Here the constraint $x_1 + x_2 \leq 6$ is also redundant: it only touches the rectangle at this corner. The corner method works as usual: $z = 0, 8, 10, 2$ at the four corners, so the optimum is $(4,2)$ with $z^* = 10$. Degenerate corners do not disturb the graphical method, but they will need care in the simplex method (Lecture 4).

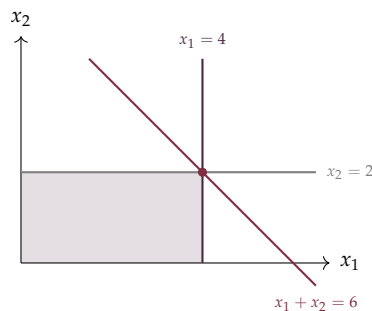


Figure 10: A degenerate corner: three lines meet at $(4,2)$.

8.3 An equality constraint

Suppose the drone workshop must use exactly all its 40 assembly hours: $x_1 + 2x_2 = 40$. The feasible plans must be *on* this line, and satisfy the other constraints. On the line, $x_1 = 40 - 2x_2$, so:

- ▶ cameras: $x_2 \leq 18$;
- ▶ testing: $3(40 - 2x_2) + 2x_2 \leq 60$, that is $120 - 4x_2 \leq 60$, so $x_2 \geq 15$.

So the feasible set is the segment from $C = (4, 18)$ to $B = (10, 15)$. Its two ends are its corners: C gives 1240 and B gives 1300, so the optimum is B . In general, an equality constraint reduces the feasible set to a part of a line: a segment, a half-line, or a single point.

9 Finding the corners without drawing

With three variables we would need a drawing in space, and with more variables no drawing is possible. So we need a way to find the corners by computation only.

Method 2.4 — Corners by algebra

A corner is a point where two constraint lines meet *and* which satisfies all the constraints. For each pair of lines: solve the system (Method 2.1), then check all the constraints.

For the drone workshop there are 5 constraints (including $x_1 \geq 0$ and $x_2 \geq 0$), so $\binom{5}{2} = 10$ pairs of lines:

Two lines		Point	Feasible?	z
assembly	testing	(10, 15)	yes	1300
assembly	cameras	(4, 18)	yes	1240
assembly	$x_1 = 0$	(0, 20)	no: 20 cameras > 18	
assembly	$x_2 = 0$	(40, 0)	no: testing 120 > 60	
testing	cameras	(8, 18)	no: assembly 44 > 40	
testing	$x_1 = 0$	(0, 30)	no: 30 cameras > 18	
testing	$x_2 = 0$	(20, 0)	yes	800
cameras	$x_1 = 0$	(0, 18)	yes	1080
cameras	$x_2 = 0$	no point (parallel lines)	—	
$x_1 = 0$	$x_2 = 0$	(0, 0)	yes	0

We find the five corners of the picture, and the best one is (10, 15). But this method becomes impossible for large problems: with n variables a corner is where n constraints meet, and the number of groups of n constraints is enormous. This is why we will need the simplex method.

Corners and slack variables. In standard form (Lecture 1, Section 9), the workshop reads

$$x_1 + 2x_2 + s_1 = 40, \quad 3x_1 + 2x_2 + s_2 = 60, \quad x_2 + s_3 = 18, \quad \text{all variables} \geq 0.$$

A constraint is binding exactly when its slack is 0, and a sign constraint $x_j \geq 0$ is binding when $x_j = 0$. At each corner two lines meet, so *two of the five variables are 0*, one for each line through the corner:

Corner	x_1	x_2	s_1	s_2	s_3
O (0, 0)	0	0	40	60	18
A (20, 0)	20	0	20	0	18
B (10, 15)	10	15	0	0	3
C (4, 18)	4	18	0	12	0
D (0, 18)	0	18	4	24	0

The simplex method uses this idea: it describes a corner by the variables that are zero.

Walking from corner to corner. Instead of computing all the corners, we can walk along the edges (Figure 11). Start at a corner, for example O ($z = 0$). Move to a *neighbouring* corner (joined by an edge) with a better value: A (800), then B (1300). At B, the two neighbours A (800) and C (1240) are worse, so we stop. Why is it correct to stop? Along each of the two edges that leave B, the value goes down (Section 4.2). Every other feasible plan is seen from B in a direction that lies between these two edges, that is, a combination with positive numbers of the two edge directions (the feasible set is convex). Since the value goes down along both edges, it also goes down in such a direction. So no feasible plan is better than B. Another path, $O \rightarrow D (1080) \rightarrow C (1240) \rightarrow B$, also works. This walk is the idea of the simplex method (Lecture 3).

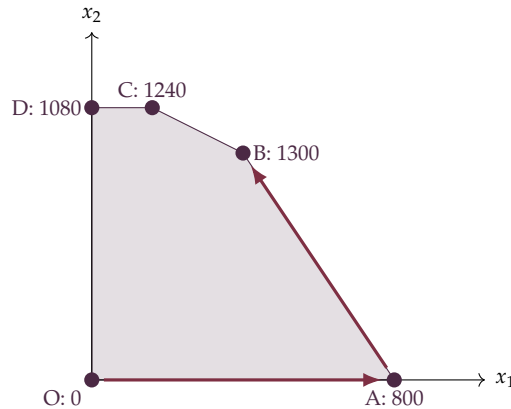


Figure 11: Walking from corner to corner: $O \rightarrow A \rightarrow B$; at B no neighbour is better.

10 What if a profit changes?

Prices change. If the profit of a drone A is c_1 thousand DA instead of 40, is the plan (10, 15) still the best? The feasible set does not change, so its corners do not change; only their values change:

$$z(O) = 0, \quad z(A) = 20c_1, \quad z(B) = 10c_1 + 900, \quad z(C) = 4c_1 + 1080, \quad z(D) = 1080.$$

B is the best corner when it is at least as good as its two neighbours A and C:

- ▶ B is at least as good as C when $10c_1 + 900 \geq 4c_1 + 1080$, that is $6c_1 \geq 180$, so $c_1 \geq 30$;
- ▶ B is at least as good as A when $10c_1 + 900 \geq 20c_1$, that is $900 \geq 10c_1$, so $c_1 \leq 90$.

(For $c_1 \geq 30$ we also have $z(B) \geq 1200 > 1080 = z(D)$, and B is always better than O.) So the plan (10, 15) stays optimal for every profit c_1 between 30 and 90 thousand DA. For $c_1 = 30$ or $c_1 = 90$, two corners tie (Case 2): this is the example of Section 7.2. Questions of this kind are the subject of Chapter 3 (sensitivity analysis).

Check yourself

Q6. Now the profit of a drone A stays at 40, and the profit of a drone B is c_2 thousand DA. For which values of c_2 is (10, 15) still an optimal plan?

11 Summary

How to recognise each case.

If you see...	then
the feasible set is empty	Case 5: infeasible
feasible points $x(t)$ with $z \rightarrow \infty$ (the line can be pushed forever)	Case 4: unbounded
the last contact is one corner	Case 1: unique optimum
two neighbouring corners give the same best value	Case 2: infinitely many optima
a line outside the set, or touching it only at a corner	redundant constraint
three or more lines through one corner	degenerate corner

What to remember.

- ▶ Tools: draw a line with two points; choose a side with a test point or with the arrow of coefficients; compute a corner by solving two equations (Method 2.1).

- ▶ Graphical method (Method 2.3): push the line of equal value in the direction of c (max) or $-c$ (min); compute the last corner touched.
- ▶ The feasible set is convex, and on a segment the value changes regularly. Therefore, if an optimum exists, a corner is optimal (Key fact 2.1): corner method (Method 2.2), corners by algebra (Method 2.4).
- ▶ To prove that a plan is optimal: multiply binding constraints by positive numbers and add them, to get " $z \leq$ its value" (or " $z \geq$ " for a minimisation).
- ▶ Five cases, three special situations; always write the answer in the form that fits the case.

What comes next. Drawing works only with two variables, and listing all the corners is too slow for large problems. In Lecture 3 we learn the simplex method, which does the "walk from corner to corner" of Section 9 with algebra, for any number of variables.

Reading. F. S. Hillier and G. J. Lieberman, *Introduction to Operations Research*, Section 3.1; H. A. Taha, *Operations Research: An Introduction*, Section 2.2.

Answers to "Check yourself"

Q1. $x_1 = 0$ gives $3x_2 = 12$, $x_2 = 4$: point $(0, 4)$. $x_2 = 0$ gives $x_1 = 12$: point $(12, 0)$. At $(0, 0)$: $0 \leq 12$ is true, so $x_1 + 3x_2 \leq 12$ keeps the side of the origin.

Q2. Add the two equations: $3x_1 = 12$, so $x_1 = 4$; then $x_2 = x_1 - 2 = 2$. Check: $8 + 2 = 10$ and $4 - 2 = 2$. The point is $(4, 2)$.

Q3. On the left: $\frac{20}{3}(2x_1 + x_2) + \frac{50}{3}(x_1 + 2x_2) = \frac{40+50}{3}x_1 + \frac{20+100}{3}x_2 = 30x_1 + 40x_2$. On the right: $\frac{20}{3} \times 8 + \frac{50}{3} \times 7 = \frac{160+350}{3} = 170$. Since the multipliers are positive, every feasible plan satisfies $30x_1 + 40x_2 \geq 170$: no purchase costs less than 170 thousand DA. The plan $(3, 2)$ costs exactly 170 thousand DA, so it is optimal. (For a minimisation with " $\geq$ " constraints, combining constraints gives a *lower* bound.)

Q4. $z = 40x_1 + 80x_2 = 40(x_1 + 2x_2)$: the lines of equal value are parallel to the assembly line. Corners: O: 0, A: 800, B: $400 + 1200 = 1600$, C: $160 + 1440 = 1600$, D: 1440. B and C tie: Case 2. All the points $(4 + 6t, 18 - 3t)$, $0 \leq t \leq 1$, are optimal, with $z^* = 1600$.

Q5. The feasible set is the region beyond the line $x_1 + x_2 = 4$; it goes to infinity. The points $(0, t)$ with $t \geq 4$ are feasible ($0 + t \geq 4$) and their value is $2t$, as large as we want. The problem is unbounded (Case 4).

Q6. The values are $z(A) = 800$, $z(B) = 400 + 15c_2$, $z(C) = 160 + 18c_2$, $z(D) = 18c_2$. B at least as good as C: $400 + 15c_2 \geq 160 + 18c_2$, so $c_2 \leq 80$. B at least as good as A: $400 + 15c_2 \geq 800$, so $c_2 \geq \frac{80}{3} \approx 26.7$. Then B is also better than D and O. So $(10, 15)$ is optimal for $\frac{80}{3} \leq c_2 \leq 80$. (For $c_2 = 80$ it ties with C: this is Q4.)