Discrete Mathematics 2

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Contents

Chapter 4: Integer Partitions

1. Introduction to Integer Partitions

Definition 1.1 (Integer Partition).
A **partition** of a non-negative integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$$

and

$$\sum_{i=1}^{k} \lambda_i := \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

The integers λ_i are called the **parts** of the partition. We write $\lambda \vdash n$ to denote that λ is a partition of n. The number of parts is k.

The number of distinct partitions of n is denoted by $p(n)$. By convention, $p(0) = 1$ (representing the empty partition, where $k = 0$).

Example.
For $n = 4$, $p(4) = 5$. The partitions are: (4), (3,1), (2,2), (2,1,1), (1,1,1,1).

Proposition 1.2.
The number of partitions of n, $p(n)$, is equal to the number of non-negative integer solutions $(m_1, m_2, \ldots, m_n)$ to the equation:

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n, \quad \text{where } m_i \geq 0 \text{ for all } i.$$

Here, m_i represents the number of times the integer i appears as a part in the partition.

Proof. Any partition $\lambda \vdash n$ can be uniquely described by specifying how many times each integer i (from 1 to n) appears as a part. Let m_i be this count. Then the sum of parts is $\sum_{i=1}^{n} i \cdot m_i = n$. Conversely, any set of non-negative integers m_i satisfying this equation corresponds to a unique partition of n. Thus, $p(n)$ is precisely the number of such solutions. □

2. Visualizing Partitions: Ferrers Diagrams

Definition 2.1 (Ferrers Diagram).
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ can be visualized using a **Ferrers diagram** (or Young diagram). It consists of k rows of boxes (or dots), where the i-th row has λ_i boxes. The rows are left-aligned.

Example.
The partition $(5, 3, 3, 1)$ of 12 can be represented.

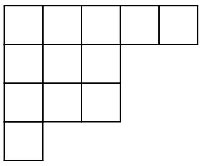

Figure 1: A representation for the partition (5,3,3,1).

Definition 2.2 (Conjugate Partition).
The **conjugate** of a partition λ, denoted λ', is the partition whose Ferrers diagram is obtained by transposing the Ferrers diagram of λ (i.e., swapping rows and columns).

Example.
The conjugate of the partition $\lambda = (5, 3, 3, 1)$ of 12 is $\lambda' = (4, 3, 3, 1, 1)$, and can be represented.

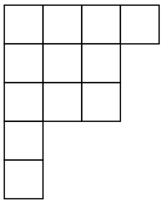

Figure 2: A representation for the partition (4,3,3,1,1).

3. Generating Functions for Partitions

The **generating function** for $p(n)$ is $P(x) = \sum_{n=0}^{\infty} p(n)x^n$, where the coefficient $p(n)$ represent the number of partitions of n.

Theorem 3.1 (Euler's Generating Function for $p(n)$)**.**

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

Proof. Consider the expansion of the infinite product

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$

as a formal power series $\sum_{n\geq 0} a_n x^n$. The coefficient a_n counts the number of times x^n appears in the expansion.
We observe that:

- The first factor $1 + x + x^2 + \cdots$ contributes a term $x^{1\cdot m_1}$ where $m_1 \geq 0$,
- The second factor $1 + x^2 + x^4 + \cdots$ contributes $x^{2\cdot m_2}$ where $m_2 \geq 0$,
- The k-th factor $1 + x^k + x^{2k} + \cdots$ contributes $x^{k\cdot m_k}$ where $m_k \geq 0$.

Thus, the exponent of x in the product is given by

$$n = 1 \cdot m_1 + 2 \cdot m_2 + \cdots + k \cdot m_k + \cdots$$

with $m_i \geq 0$ for all i. Crucially, for a fixed n, we must have $m_i = 0$ for all $i > n$ because if $i > n$ and $m_i \geq 1$, then $i \cdot m_i \geq i > n$, which would make the sum exceed n. Therefore, only finitely many terms contribute to each x^n.
The coefficient a_n is the number of solutions $(m_1, m_2, \dots)$ to this equation where the sum is finite.
Each solution corresponds uniquely to a partition of n:

- m_1 counts the number of parts of size 1,
- m_2 counts the number of parts of size 2,
- $\vdots$
- m_k counts the number of parts of size k.

Since this enumerates all partitions of n, we have $a_n = p(n)$. Therefore,

$$\sum_{n\geq 0} p(n)x^n = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}.$$

□

4. Restricted Partitions and Euler's Partition Theorem

4.1 Partitions into Exactly k Parts

Definition 4.1.
Let $p(n, k)$ denote the number of partitions of n into exactly k parts.

Example.
For $n = 5, k = 2$: partitions are $(4, 1), (3, 2)$. So $p(5, 2) = 2$.

Remark.
$p(n, 1) = 1$ (partition (n)); $p(n, n) = 1$ (partition $(1, \ldots, 1)$); $p(n, k) = 0$ if $n < k$; $p(0, 0) = 1$.

Proposition 4.2 (Recurrence for $p(n, k)$)**.**
For $n, k \geq 1$: $p(n, k) = p(n-k, k-1)+p(n-k, k) = p(n-1, k-1)+p(n-k, k) = \sum_{i=1}^{min(k,n-k)} p(n-k, i)$.

Proof. Consider partitions of n into k parts:

1. Smallest part is 1: Remove it → partition of $n - 1$ into $k - 1$ parts ($p(n - 1, k - 1)$ ways).
2. Smallest part is > 1: Subtract 1 from each of the k parts → partition of $n - k$ into k parts ($p(n - k, k)$ ways).

These cases are disjoint and cover all possibilities. □

The table below lists $p(n, k)$:

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0
5	0	1	2	2	1	1	0	0	0
6	0	1	3	3	2	1	1	0	0
7	0	1	3	4	3	2	1	1	0
8	0	1	4	5	5	3	2	1	1

Theorem 4.3.
The number of partitions of n into k parts is equal to the number of partitions of n where the largest part is k.

Proof. Let λ be a partition of n into k parts (its Ferrers diagram has k rows). The largest part of its conjugate λ' is equal to the number of rows in the diagram of λ. Thus, the largest part of λ' is k. This mapping is a bijection. □

4.1.1 Partitions into at most k Parts

Definition 4.4.
Let $p_{\leq k}(n)$ denote the number "partitions of n with at most k parts."

Example.
Let $n = 5$ and $k = 2$. We are looking for partitions of 5 with at most 2 parts.

- Partitions with exactly 1 part ($j = 1$): (5). There is $p(5, 1) = 1$ such partition.
- Partitions with exactly 2 parts ($j = 2$): (4,1), (3,2). There are $p(5, 2) = 2$ such partitions.

So, $p_{\leq 2}(5) = p(5, 1) + p(5, 2) = 1 + 2 = 3$.

Proposition 4.5.
The function $p_{\leq k}(n)$ (counting partitions of n into at most k parts) has the following properties:

1. For $n \geq 0$:
$$p_{\leq k}(n) = \sum_{j=0}^{k} p(n, j)$$
2. For $k \geq 1$ and $n \geq 0$:
$$p(n, k) = p_{\leq k}(n) - p_{\leq (k-1)}(n)$$

3. If $k \geq n$:

$$p_{\leq k}(n) = p(n)$$

Proof.

1. By definition, $p_{\leq k}(n)$ counts partitions of n with j parts where $0 \leq j \leq k$. Since partitions with different numbers of parts are disjoint, we sum the exact counts: $p_{\leq k}(n) = \sum_{r=0}^{k} p(n, j)$.
2. The set of partitions with $\leq k$ parts decomposes as:

$$\{\text{partitions with } \leq k \text{ parts}\} = \{\text{exactly } k \text{ parts}\} \sqcup \{\leq k-1 \text{ parts}\}$$

Thus $p_{\leq k}(n) = p(n, k) + p_{\leq k-1}(n)$, and rearranging gives the identity.

3. For $n \geq 1$, any partition of n has at most n parts (since each part is ≥ 1). When $k \geq n$, the condition "$\leq k$ parts" is redundant, so $p_{\leq k}(n) = p(n)$. For $n = 0$, both sides equal 1 (empty partition).

here $\square$

4.1.2 Generating function of $p_{\leq k}(n)$ and $p(n, k)$

Theorem 4.6 (Generating Functions by Number of Parts).
Let $p_{\leq k}(n)$ be the number of partitions of n into at most k positive parts, and let $p(n, k)$ be the number of partitions of n into exactly k positive parts. Their respective ordinary generating functions are:

1. For partitions into at most k parts:

$$P_{\leq k}(x) = \sum_{n=0}^{\infty} p_{\leq k}(n) x^n = \prod_{i=1}^{k} \frac{1}{1 - x^i}$$

2. For partitions into exactly k parts:

$$P_k(x) = \sum_{n=k}^{\infty} p(n, k) x^n = \prod_{i=1}^{k} \frac{x^k}{1 - x^i}$$

(The sum for $P_k(x)$ starts at $n = k$ as n must be at least k to have k positive parts.)

Proof.

1. By conjugation of Ferrers diagrams (transposition), the number of partitions of n with at most k parts equals the number of partitions of n with largest part $\leq k$. The generating function for partitions with largest part $\leq k$ is:

$$\begin{aligned}(1 + x^1 + x^2 + \cdots)(1 + x^2 + x^{2\times 2} + \cdots) \times \cdots \times (1 + x^k + x^{2k} + \cdots) &= \prod_{i=1}^{k} \left(1 + x^i + x^{2i} + \cdots\right) \\ &= \prod_{i=1}^{k} \frac{1}{1 - x^i}\end{aligned}$$

since we may use any part size $i \in \{1, \ldots, k\}$ with unlimited multiplicity. Thus:

$$P_{\leq k}(x) = \sum_{n=0}^{\infty} p_{\leq k}(n) x^n = \prod_{i=1}^{k} \frac{1}{1 - x^i}.$$

2. Using the combinatorial identity $p(n, k) = p_{\leq k}(n) - p_{\leq k-1}(n)$, the generating function for exactly k parts is:

$$P_k(x) = P_{\leq k}(x) - P_{\leq k-1}(x) = \prod_{i=1}^{k} \frac{1}{1 - x^i} - \prod_{i=1}^{k-1} \frac{1}{1 - x^i}.$$

Factoring out $\prod_{i=1}^{k-1} \frac{1}{1-x^i}$ gives:

$$P_k(x) = \left(\prod_{i=1}^{k-1} \frac{1}{1 - x^i}\right)\left(\frac{1}{1 - x^k} - 1\right) = \left(\prod_{i=1}^{k-1} \frac{1}{1 - x^i}\right) \frac{x^k}{1 - x^k} = \prod_{i=1}^{k} \frac{x^k}{1 - x^i}.$$

The sum starts at $n = k$ since partitions require at least k elements when using k positive parts.

□

4.2 General Approach to Restricted Partitions

Generating functions can be adapted to count partitions with various restrictions. If a part j can appear s_1 or s_2 or ... s_m times, its factor in the generating function is $(x^{js_1} + x^{js_2} + \cdots + x^{js_m})$. The overall generating function is the product of these factors.

4.2.1 Partitions into Distinct Parts and Odd Parts

Example (Generating Function for Partitions into Distinct Parts).
Let $p_d(n)$ be the number of partitions of n where all parts are distinct. Each part k can appear 0 or 1 time. Factor for part k: $(1 + x^k)$. Generating function: $P_d(x) = \prod_{k=1}^{\infty}(1 + x^k)$.

Example (Generating Function for Partitions into Odd Parts).
Let $p_o(n)$ be the number of partitions of n where all parts are odd. Only odd parts $j = 2k - 1$ are allowed, appearing any number of times. Factor for odd part j: $\frac{1}{1-x^j}$. Generating function: $P_o(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$.

Theorem 4.7 (Euler's Partition Theorem, 1748).
The number of partitions of n into distinct parts equals the number of partitions of n into odd parts: $p_d(n) = p_o(n)$ for all $n \geq 0$.

Proof. We show $P_d(x) = P_o(x)$.

$$\begin{aligned} P_d(x) &= \prod_{k=1}^{\infty}(1 + x^k) = \prod_{k=1}^{\infty} \frac{1 - x^{2k}}{1 - x^k} \\ &= \frac{\prod_{j=1}^{\infty}(1 - x^{2j})}{\left(\prod_{j=1}^{\infty}(1 - x^{2j-1})\right)\left(\prod_{j=1}^{\infty}(1 - x^{2j})\right)} = \frac{1}{\prod_{j=1}^{\infty}(1 - x^{2j-1})} = P_o(x). \end{aligned}$$

□

4.2.2 Further Examples of Restricted Partitions

Example (Partitions where the part '2' appears at most once).
Let $q(n)$ be the number of such partitions. Generating function: $Q(x) = (1 + x^2)\prod_{k\neq 2, k\geq 1} \frac{1}{1-x^k} = (1 - x^4)P(x)$. So, $q(n) = p(n) - p(n - 4)$ (with $p(j) = 0$ if $j < 0$).

Example (Partitions where each part appears at most twice).
Let $r(n)$ be the number of such partitions. Factor for part k: $(1 + x^k + x^{2k})$. Generating function: $R(x) = \prod_{k=1}^{\infty}(1 + x^k + x^{2k}) = \prod_{k=1}^{\infty} \frac{1-x^{3k}}{1-x^k} = \frac{P(x)}{P(x^3)}$.

5. Exercises

Exercise.
List partitions of $n = 6$. Verify $p_d(6) = p_o(6)$. Calculate $p(6, 3)$ using its recurrence.

Exercise.
Draw the Ferrers diagram for $(5, 2, 2)$ and find its conjugate.

Exercise.
Write the generating function for partitions where all parts are ≥ 2 and distinct.