Discrete Mathematics 2

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Contents

Chapter 3: Other Important Combinatorial Sequences

1. Eulerian Numbers

Definition 1.1.
Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n] = \{1, 2, \ldots, n\}$.

- An **ascent** is a position i, $1 \le i \le n-1$, such that $\pi_i < \pi_{i+1}$.
- A **descent** is a position i, $1 \le i \le n-1$, such that $\pi_i > \pi_{i+1}$.

Example.
For $\pi = 3142$, there is one ascent, at position 2, and two descents, at positions 1 and 3:

$$3 > 1 < 4 > 2.$$

For 1234, there are 3 ascents and no descents. For 4321, there are no ascents and 3 descents.

Proposition 1.2.
Every permutation of $[n]$ has

$$\text{number of ascents} + \text{number of descents} = n - 1.$$

Proof. There are exactly $n-1$ adjacent pairs (π_i, π_{i+1}). Since all entries are distinct, each pair is either an ascent or a descent, and never both. □

Definition 1.3.
The **Eulerian number** $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $[n]$ with exactly k ascents.

Example.
For $n = 3$, the six permutations are

$$123, 132, 213, 231, 312, 321.$$

Hence

$$\left\langle {3 \atop 0} \right\rangle = 1, \qquad \left\langle {3 \atop 1} \right\rangle = 4, \qquad \left\langle {3 \atop 2} \right\rangle = 1.$$

Theorem 1.4 (Recurrence for Eulerian Numbers)**.**
For $n \ge 1$,

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle,$$

with $\left\langle {0 \atop 0} \right\rangle = 1$ and $\left\langle {n \atop k} \right\rangle = 0$ if $k < 0$ or $k \ge n$.

Proof. We insert the largest element n into a permutation of $[n-1]$.
If the old permutation has k ascents, then we must insert n without increasing the number of ascents. This is possible in $k+1$ positions: after an ascent, or at the beginning. This gives

$$(k+1)\left\langle {n-1 \atop k} \right\rangle.$$

If the old permutation has $k-1$ ascents, then we must create one new ascent. This is possible in $n-k$ positions: after a descent, or at the end. This gives

$$(n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

Adding the two cases proves the recurrence. □

1.1 Basic Properties

For $n \ge 1$,

$$\left\langle {n \atop 0} \right\rangle = 1, \qquad \left\langle {n \atop n-1} \right\rangle = 1,$$

because the only permutation with no ascents is the decreasing one, and the only permutation with $n-1$ ascents is the increasing one.

Theorem 1.5 (Sum Identity).
For every integer $n \geq 1$,

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n!.$$

Proof. Every permutation of $[n]$ has exactly one possible number of ascents, between 0 and $n-1$. Therefore the sum counts all permutations of $[n]$, namely $n!$. □

Remark.
The symmetry

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$$

can be proved by reversing the order of each permutation. This changes ascents into descents.

1.2 A Small Eulerian Triangle

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	0				
2	1	1	0			
3	1	4	1	0		
4	1	11	11	1	0	
5	1	26	66	26	1	0

2. Harmonic Numbers

Definition 2.1.
The **harmonic numbers** are defined by

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad (n \geq 1).$$

Proposition 2.2.
For $n \geq 1$,

$$H_n = H_{n-1} + \frac{1}{n}.$$

Proof. This follows immediately from

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} = H_{n-1} + \frac{1}{n}.$$

□

Theorem 2.3 (Generating Function).
The ordinary generating function of the harmonic numbers is

$$\sum_{n \geq 1} H_n x^n = \frac{-\ln(1-x)}{1-x}.$$

Proof. Since

$$-\ln(1-x) = \sum_{j \geq 1} \frac{x^j}{j} \qquad \text{and} \qquad \frac{1}{1-x} = \sum_{r \geq 0} x^r,$$

their product is

$$\left(\sum_{j\geq 1}\frac{x^j}{j}\right)\left(\sum_{r\geq 0}x^r\right)=\sum_{n\geq 1}\left(\sum_{j=1}^{n}\frac{1}{j}\right)x^n=\sum_{n\geq 1}H_nx^n.$$

□

3. Catalan and Delannoy Numbers

3.1 Catalan Numbers

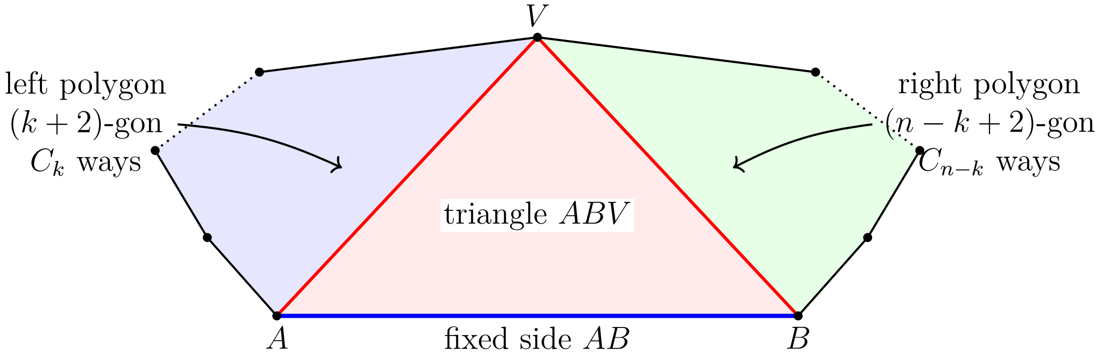
Definition 3.1.
The **Catalan number** C_n is the number of triangulations of a convex polygon with $n+2$ vertices. We set $C_0=1$.

Example.
$C_1=1$, since a triangle has one triangulation. Also $C_2=2$, since a quadrilateral has two possible diagonals.

Theorem 3.2 (Catalan Recurrence).
For every $n\geq 0$,

$$C_{n+1}=\sum_{k=0}^{n}C_kC_{n-k}.$$

Proof. We want to count the number of triangulations of a convex polygon with $n+3$ sides. By definition, this number is C_{n+1}.
Fix one side of the polygon, and call it AB. In any triangulation, the side AB belongs to exactly one triangle. Let V be the third vertex of this triangle.
Then the two segments AV and BV divide the polygon into: the triangle ABV, a polygon on the left, and a polygon on the right.



Suppose the polygon on the left has $k+2$ sides, where $0\leq k\leq n$. Then, by the definition of Catalan numbers, the number of triangulations of this left polygon is C_k.
Now let us find the number of sides of the polygon on the right. The original polygon has $n+3$ sides. The side AB is used as the fixed side, and the segments AV and BV become sides of the two smaller polygons. Hence, if the right polygon has r sides, then

$$(k+2)+r=(n+3)-1+2=n+4.$$

Therefore,

$$r=n-k+2.$$

So the right polygon has $n-k+2$ sides, and the number of its triangulations is

$$C_{n-k}.$$

For this fixed choice of the vertex V, the triangulation of the left polygon and the triangulation of the right polygon are independent. Therefore, by the multiplication rule, the total number of triangulations in this case is

$$C_k\,C_{n-k}.$$

Now the vertex V can move along the boundary from one end to the other. When V changes, the value of k takes all values from 0 to n. So, by the addition rule, the total number of triangulations is

$$\sum_{k=0}^{n} C_k\, C_{n-k}.$$

But this total is exactly the number of triangulations of the $(n+3)$-gon, which is C_{n+1}. Hence,

$$C_{n+1} = \sum_{k=0}^{n} C_k\, C_{n-k}.$$

We use the convention $C_0 = 1$. This is a convention, not the number of triangulations of a polygon. It is used for the extreme cases $k = 0$ or $k = n$, when one side of the triangle ABV leaves no polygonal part on that side. □

Remark (Other interpretations).
The Catalan number C_n also counts:

- full parenthesizations of a product of $n+1$ factors;
- rooted binary trees with n vertices;
- Dyck paths with n up-steps and n down-steps which never go below the horizontal axis.

Theorem 3.3 (Generating Function).
Let

$$C(x) = \sum_{n\geq 0} C_n x^n.$$

Then

$$C(x) = 1 + xC(x)^2.$$

Consequently,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Proof. Using $C_0 = 1$ and the recurrence,

$$C(x) = 1 + \sum_{n\geq 0} C_{n+1}x^{n+1} = 1 + x\sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right)x^n.$$

The last sum is the Cauchy product $C(x)^2$. Hence $C(x) = 1 + xC(x)^2$. Solving the quadratic equation gives the stated formula, and the sign is chosen so that $C(0) = 1$. □

Theorem 3.4 (Closed Formula).
For every $n \geq 0$,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Starting from the generating function in Theorem 3.3:

$$C(x) = \frac{1-\sqrt{1-4x}}{2x},$$

we expand $\sqrt{1-4x}$ using the generalized binomial theorem:

$$\begin{aligned}
\sqrt{1-4x} = (1-4x)^{1/2} &= \sum_{k=0}^{\infty}\binom{1/2}{k}(-4x)^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!}(-4x)^k \quad \text{(see Lemma below)} \\
&= 1 - 2\sum_{k=1}^{\infty}\frac{(2k-2)!}{k!(k-1)!}x^k.
\end{aligned}$$

Substitute this back into $C(x)$:

$$C(x) = \frac{1 - \left(1 - 2\sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!} x^k\right)}{2x}$$
$$= \frac{2\sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!} x^k}{2x}$$
$$= \sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!} x^{k-1}.$$

Reindex with $n = k - 1$:

$$C(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Comparing coefficients of x^n, we obtain:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

□

3.2 Delannoy Numbers

Definition 3.5.
The **Delannoy number** $D(m, n)$ counts the number of paths from $(0, 0)$ to (m, n) using the steps

$$E = (1, 0), \qquad N = (0, 1), \qquad NE = (1, 1).$$

Example.
From $(0, 0)$ to $(1, 1)$, there are three paths:

$$E, N; \qquad N, E; \qquad NE.$$

Thus $D(1, 1) = 3$.

Proposition 3.6 (Recurrence)**.**
For $m, n > 0$,

$$D(m, n) = D(m-1, n) + D(m, n-1) + D(m-1, n-1),$$

with boundary conditions

$$D(m, 0) = 1, \qquad D(0, n) = 1.$$

Proof. Look at the last step of a path ending at (m, n). It is either:

- an east step from $(m-1, n)$;
- a north step from $(m, n-1)$;
- a northeast step from $(m-1, n-1)$.

These cases are disjoint and cover all possibilities. □

Example.
The first values are

$D(m, n)$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$m = 0$	1	1	1	1	1
$m = 1$	1	3	5	7	9
$m = 2$	1	5	13	25	41
$m = 3$	1	7	25	63	129

Proposition 3.7 (Binomial Formula)**.**
For $m, n \geq 0$,

$$D(m, n) = \sum_{j=0}^{\min(m,n)} \binom{m+n-j}{m} \binom{m}{j}.$$

Proof. We count the paths from $(0,0)$ to (m,n) using the steps

$$E = (1,0), \qquad N = (0,1), \qquad NE = (1,1).$$

Fix j, the number of NE-steps.

Then the path has $m-j$ steps of type E, $n-j$ steps of type N, and j steps of type NE. Hence the total number of steps is

$$(m-j) + (n-j) + j = m + n - j.$$

First, choose the positions of the steps which have an East component. These are the E-steps and the NE-steps. There are exactly m such positions, so this can be done in

$$\binom{m+n-j}{m}$$

ways.

Second, among these m positions, choose the j positions which are NE-steps. The remaining chosen positions are E-steps. This can be done in

$$\binom{m}{j}$$

ways.

All the other positions are forced to be N-steps. Therefore, the number of paths with exactly j steps of type NE is

$$\binom{m+n-j}{m}\binom{m}{j}.$$

Finally, j can take all values from 0 to $\min(m,n)$. Thus

$$D(m,n) = \sum_{j=0}^{\min(m,n)} \binom{m+n-j}{m}\binom{m}{j}.$$

□