Discrete Mathematics 2

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Contents

Chapter 2: Stirling Numbers and Related Sequences

1. Introduction

This chapter focuses on a fascinating family of such numbers: Stirling numbers of the first and second kind, Lah numbers, and Bell numbers. These numbers arise naturally in problems involving permutations, partitions, and lists.

1.1 Factorial functions

Definition 1.1 (Falling and Rising Factorial Functions).
Falling Factorial: For a non-negative integer n and a real or complex number x, the falling factorial, denoted by $(x)_n$, is defined as:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

with the convention that $(x)_0 = 1$.
Rising Factorial: For a non-negative integer n and a real or complex number x, the rising factorial (or Pochhammer symbol), denoted by $(x)^{(n)}$, is defined as:

$$(x)^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$$

with the convention that $(x)^{(0)} = 1$.

1.1.1 Properties of Falling and Rising Factorials

- **Relation to factorials**:
  $(n)_n = n!$ (by definition, since $n(n-1)\cdots 1 = n!$), $1^{(n)} = n!$ (rising factorial $1(2)(3)\cdots n = n!$).
- **Symmetry**:
$$(x)_n = (-1)^n(-x)^{(n)}, \quad x^{(n)} = (-1)^n(-x)_n.$$
- **Connection to binomial coefficients**: For $x \in \mathbb{C}$,
$$\binom{x}{k} = \frac{(x)_k}{k!}, \quad \binom{-x}{k} = (-1)^k \frac{x^{(k)}}{k!}.$$
- **Recurrence relations**:
$$(x)_{n+1} = (x-n)(x)_n, \quad x^{(n+1)} = (x+n)x^{(n)}.$$

1.2 Permutations and cycles

A **permutation** π on n elements is a bijective function:

$$\pi : [n] \to [n], \quad \text{where } [n] = \{1, 2, \ldots, n\}.$$

Permutations can be represented in two standard forms:

One-Line Notation

The permutation π is written as a sequence of its outputs:

$$\pi = \pi(1)\ \pi(2)\ \pi(3)\ \cdots\ \pi(n).$$

- The i-th position represents $\pi(i)$.
- Example: For $n = 4$, $\pi = 3\ 1\ 4\ 2$ means:
$$\pi(1) = 3,\ \pi(2) = 1,\ \pi(3) = 4,\ \pi(4) = 2.$$

Two-Line Notation

The permutation π is displayed as a matrix with two rows:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

- Top row: Domain elements.
- Bottom row: Corresponding images under π.
- Example for $\pi = 3\ 1\ 4\ 2$:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

1.2.1 Cycle Decomposition

Cycle Structure

A permutation decomposes into disjoint **cycles**, which are closed chains of mappings $i \to \pi(i)$. For example, consider the permutation π in one-line notation:

$$\pi = 6\ 5\ 2\ 7\ 1\ 3\ 4\ 8.$$

In two-line notation, this becomes:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 7 & 1 & 3 & 4 & 8 \end{pmatrix}.$$

Tracing the mappings $i \to \pi(i)$, we observe the following cycles:

$$1 \to 6 \to 3 \to 2 \to 5 \to 1, \qquad 4 \to 7 \to 4, \qquad 8 \to 8.$$

- Each cycle is a closed loop of distinct elements.
- Fixed points (e.g., 8) form 1-cycles .

Cycle Notation

The permutation is written as a product of cycles:

$$\pi = (1\ 6\ 3\ 2\ 5)(4\ 7)(8).$$

Equivalence: Cycles can be reordered or rotated:

$$\pi = (8)(7\ 4)(3\ 2\ 5\ 1\ 6) = (6\ 3\ 2\ 5\ 1)(4\ 7).$$

2. Stirling Numbers

2.1 Stirling Numbers of the First Kind

Definition 2.1 (Stirling numbers of the first kind).
For non-negative integers n and k with $0 \le k \le n$, the **Stirling number of the first kind**, denoted $\left[{n \atop k}\right]$, counts:

- The number of ways to seat n people around k non-empty, identical circular tables (rotations of a table do not count as distinct arrangements).
- Equivalently, it counts the number of permutations of n elements that decompose into exactly k disjoint cycles.

Examples.

- For $n = 4$ and $k = 2$, the Stirling number $\left[{4 \atop 2}\right] = 11$ corresponds to:

  Cycle Type (3,1) - 8 permutations

$$\begin{aligned} &(1\ 2\ 3)(4), \quad (1\ 3\ 2)(4), \quad (1\ 2\ 4)(3), \quad (1\ 4\ 2)(3), \\ &(1\ 3\ 4)(2), \quad (1\ 4\ 3)(2), \quad (2\ 3\ 4)(1), \quad (2\ 4\ 3)(1). \end{aligned}$$

Cycle Type (2,2) - 3 permutations

$$(1\ 2)(3\ 4), \quad (1\ 3)(2\ 4), \quad (1\ 4)(2\ 3).$$

Hence, $\begin{bmatrix}4\\2\end{bmatrix} = 11$.

- For $n = 3$, the permutations of [3] and their cycle decompositions:
  - $123 \leftrightarrow (1)(2)(3)$ (3 cycles)
  - $132 \leftrightarrow (1)(2\,3)$ (2 cycles)
  - $213 \leftrightarrow (1\,2)(3)$ (2 cycles)
  - $231 \leftrightarrow (1\,2\,3)$ (1 cycle)
  - $312 \leftrightarrow (1\,3\,2)$ (1 cycle)
  - $321 \leftrightarrow (1\,3)(2)$ (2 cycles)

$$\begin{bmatrix}3\\1\end{bmatrix} = 2, \quad \begin{bmatrix}3\\2\end{bmatrix} = 3, \quad \begin{bmatrix}3\\3\end{bmatrix} = 1.$$

This counts: 2 ways to seat at 1 table, 3 ways for 2 tables, 1 way for 3 tables.

2.1.1 Particular Values of Unsigned Stirling Numbers of the First Kind

The unsigned Stirling numbers of the first kind, $\begin{bmatrix}n\\k\end{bmatrix}$, satisfy the following special cases:

- $\begin{bmatrix}n\\0\end{bmatrix} = 0$ for $n \geq 1$ (no ways to partition n elements into 0 cycles), and $\begin{bmatrix}0\\0\end{bmatrix} = 1$ (empty permutation).
- $\begin{bmatrix}n\\n\end{bmatrix} = 1$ (each element forms its own 1-cycle).
- $\begin{bmatrix}n\\1\end{bmatrix} = (n-1)!$ (number of cyclic permutations of n elements).
- $\begin{bmatrix}n\\n-1\end{bmatrix} = \binom{n}{2}$ (one 2-cycle and $n-2$ fixed points).

2.1.2 Recurrence of $\begin{bmatrix}n\\k\end{bmatrix}$

Theorem 2.2 (Recurrence for Stirling Numbers of the First Kind)**.**
For integers $n, k \geq 1$, the following recurrence relation holds:

$$\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n-1\\k-1\end{bmatrix} + (n-1)\begin{bmatrix}n-1\\k\end{bmatrix}$$

with base cases $\begin{bmatrix}n\\0\end{bmatrix} = 0$ for $n \geq 1$, $\begin{bmatrix}0\\0\end{bmatrix} = 1$, and $\begin{bmatrix}n\\k\end{bmatrix} = 0$ if $k > n$.

2.1.3 Triangle of Stirling Numbers of the First Kind

Using the recurrence $\begin{bmatrix}n\\k\end{bmatrix} = (n-1)\begin{bmatrix}n-1\\k\end{bmatrix} + \begin{bmatrix}n-1\\k-1\end{bmatrix}$, the triangle is:

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

2.2 Stirling Numbers of the Second Kind

Definition 2.3 (Stirling numbers of the second kind)**.**
For non-negative integers n and k with $0 \le k \le n$, the **Stirling number of the second kind**, denoted $\left\{ {n \atop k} \right\}$ or $S(n,k)$, counts:

- The number of ways to partition a set of n distinct (labelled) elements into k non-empty, indistinguishable (unlabelled) subsets.
- It also counts the number of ways to place n distinct balls into k identical non-empty boxes.

Examples.

- For $n = 4$ and $k = 2$, the Stirling number $\left\{ {4 \atop 2} \right\} = 7$ corresponds to the ways to partition the set $\{1,2,3,4\}$ into 2 non-empty subsets:

  Partition Type {3,1} - $\binom{4}{3} = 4$ **partitions**

  $$\{1,2,3\},\{4\}; \quad \{1,2,4\},\{3\}; \quad \{1,3,4\},\{2\}; \quad \{2,3,4\},\{1\}.$$

  Partition Type {2,2} - $\frac{1}{2!}\binom{4}{2} = 3$ **partitions**

  $$\{1,2\},\{3,4\}; \quad \{1,3\},\{2,4\}; \quad \{1,4\},\{2,3\}.$$

  Hence, $\left\{ {4 \atop 2} \right\} = 4 + 3 = 7$.
- For $n = 3$, the partitions of $\{1,2,3\}$:
  - Into 1 subset ($k = 1$): $\{1,2,3\}$ (1 way)
  - Into 2 subsets ($k = 2$): $\{1,2\},\{3\}$; $\{1,3\},\{2\}$; $\{2,3\},\{1\}$ (3 ways)
  - Into 3 subsets ($k = 3$): $\{1\},\{2\},\{3\}$ (1 way)

  Thus,
  $$\left\{ {3 \atop 1} \right\} = 1, \quad \left\{ {3 \atop 2} \right\} = 3, \quad \left\{ {3 \atop 3} \right\} = 1.$$

2.2.1 Particular Values of Stirling Numbers of the Second Kind

The Stirling numbers of the second kind, $\left\{ {n \atop k} \right\}$, satisfy the following special cases:

- $\left\{ {n \atop 0} \right\} = 0$ for $n \ge 1$ (no ways to partition n elements into 0 non-empty subsets), and $\left\{ {0 \atop 0} \right\} = 1$ (empty set has one partition: into 0 subsets).
- $\left\{ {n \atop n} \right\} = 1$ (each element must be in its own subset).
- $\left\{ {n \atop 1} \right\} = 1$ (all elements must be in the single subset).
- $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$ (one subset must contain exactly two elements, the rest are singletons; choose the pair in $\binom{n}{2}$ ways).
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$ (Each element goes into set A or set B, 2^n ways. Exclude cases where A or B is empty (2 ways). Since sets A, B are indistinguishable, divide by 2: $(2^n - 2)/2 = 2^{n-1} - 1$).

2.2.2 Recurrence of $\left\{ {n \atop k} \right\}$

Theorem 2.4 (Recurrence for Stirling Numbers of the Second Kind)**.**
For integers $n, k \ge 1$, the following recurrence relation holds:
$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$$
with base cases $\left\{ {n \atop 0} \right\} = 0$ for $n \ge 1$, $\left\{ {0 \atop 0} \right\} = 1$, and $\left\{ {n \atop k} \right\} = 0$ if $k > n$.

2.2.3 Triangle of Stirling Numbers of the Second Kind

Using the recurrence $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$, the triangle is:

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	1	3	1					
4	0	1	7	6	1				
5	0	1	15	25	10	1			
6	0	1	31	90	65	15	1		
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

2.3 Lah Numbers (Stirling Numbers of the Third Kind)

Definition 2.5 (Lah Numbers)**.**
For non-negative integers n and k with $0 \le k \le n$, the **Lah number**, denoted $\left\lfloor {n \atop k} \right\rfloor$, counts:

- The number of ways to partition n distinct (labeled) elements into k non-empty, indistinguishable (unlabeled) linearly ordered subsets (i.e., sequences where the order within each subset matters).
- Equivalently, it counts the number of ways to arrange n elements into k non-empty lists where the order of elements within each list matters, but the order of the lists themselves does not.

Examples.

- For $n = 3$ and $k = 2$, the Lah number $\left\lfloor {3 \atop 2} \right\rfloor = 6$ corresponds to the ways to partition $\{1, 2, 3\}$ into 2 ordered lists:
$$\{[1], [2, 3]\}, \quad \{[1], [3, 2]\}, \quad \{[2], [1, 3]\}, \quad \{[2], [3, 1]\}, \quad \{[3], [1, 2]\}, \quad \{[3], [2, 1]\}.$$
- For $n = 4$ and $k = 2$, the Lah number $\left\lfloor {4 \atop 2} \right\rfloor = 36$. This counts all partitions into two ordered lists, such as $[1, 2, 3], [4]$ or $[1, 4], [2, 3]$, where internal order matters in each list.
- For $n = 2$, the partitions of $\{1, 2\}$:
  - Into 1 subset ($k = 1$): $[1, 2]$ or $[2, 1]$ (2 ways, but indistinguishable as a single list). However, since the list order matters, $\left\lfloor {2 \atop 1} \right\rfloor = 2$.
  - Into 2 subsets ($k = 2$): $\{[1], [2]\}$ (1 way, as singleton order is fixed).

  Thus,
$$\left\lfloor {2 \atop 1} \right\rfloor = 2, \quad \left\lfloor {2 \atop 2} \right\rfloor = 1.$$

2.3.1 Particular Values of Lah Numbers

The Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ satisfy the following special cases:

- $\left\lfloor {n \atop 0} \right\rfloor = 0$ for $n \ge 1$, and $\left\lfloor {0 \atop 0} \right\rfloor = 1$ (analogous to Stirling numbers).
- $\left\lfloor {n \atop n} \right\rfloor = 1$ (each element forms its own singleton list; order is irrelevant).
- $\left\lfloor {n \atop 1} \right\rfloor = n!$ (all elements form a single ordered list, equivalent to permutations of n elements).
- $\left\lfloor {n \atop n-1} \right\rfloor = 2\binom{n}{2}$ (one list contains two elements in order, and the remaining $n - 2$ are singletons; choose the pair in $\binom{n}{2}$ ways and order them in 2 ways).
- $\left\lfloor {n \atop k} \right\rfloor = 0$ if $k > n$ or $k < 0$.

2.3.2 Recurrence of $\left\lfloor {n \atop k} \right\rfloor$

Theorem 2.6 (Recurrence for Lah Numbers)**.**
For integers $n, k \geq 1$, the following recurrence relation holds:

$$\left\lfloor {n \atop k} \right\rfloor = \left\lfloor {n-1 \atop k-1} \right\rfloor + (n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor$$

with base cases $\left\lfloor {n \atop 0} \right\rfloor = 0$ for $n \geq 1$, $\left\lfloor {0 \atop 0} \right\rfloor = 1$, and $\left\lfloor {n \atop k} \right\rfloor = 0$ if $k > n$.

2.3.3 Triangle of Lah Numbers

Using the recurrence $\left\lfloor {n \atop k} \right\rfloor = (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor + \left\lfloor {n-1 \atop k-1} \right\rfloor$, the triangle is:

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	2	1							
3	0	6	6	1						
4	0	24	36	12	1					
5	0	120	240	120	20	1				
6	0	720	1800	1200	300	30	1			
7	0	5040	15120	12600	4200	630	42	1		
8	0	40320	141120	141120	58800	11760	1176	56	1	
9	0	362880	1451520	1693440	846720	211680	28224	2016	72	1

Theorem 2.7.
The Lah numbers, denoted $\left\lfloor {n \atop k} \right\rfloor$, which count the number of ways to partition a set of n elements into k non-empty ordered lists, satisfy the identity:

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1}.$$

2.4 Stirling Numbers, Lah Numbers, and Factorial Functions

Theorem 2.8 (Factorial Expansions)**.**
For non-negative integers n, k $(0 \leq k \leq n)$ and a real or complex variable x:

$$\textbf{(1)} \quad x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \text{(Stirling numbers of the first kind)},$$

$$\textbf{(2)} \quad x^{n} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k \quad \text{(Stirling numbers of the second kind)},$$

$$\textbf{(3)} \quad x^{(n)} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor (x)_k \quad \text{(Lah numbers)},$$

where:

- Falling factorial: $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$
- Rising factorial: $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$

Examples.

Examples for $n = 2$:

- **First Kind Identity:**

$$x^{(2)} = x(x+1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 2 \end{bmatrix} x^2 = 1 \cdot x + 1 \cdot x^2 = x + x^2$$

  Verified: $x(x+1) = x^2 + x$.

- **Second Kind Identity:**

$$x^2 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} (x)_1 + \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} (x)_2 = 1 \cdot x + 1 \cdot x(x-1) = x + x^2 - x = x^2$$

  Verified: $x^2 = x^2$.

- **Lah Numbers Identity:**

$$x^{(2)} = \left\lfloor {2 \atop 1} \right\rfloor (x)_1 + \left\lfloor {2 \atop 2} \right\rfloor (x)_2 = 2x + 1 \cdot x(x-1) = 2x + x^2 - x = x^2 + x$$

  Verified: $x(x+1) = x^2 + x$.

Examples for $n = 3$:

- **First Kind Identity:**

$$x^{(3)} = x(x+1)(x+2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \end{bmatrix} x^2 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} x^3 = 2x + 3x^2 + 1x^3 = x^3 + 3x^2 + 2x$$

  Verified: $x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

- **Second Kind Identity:**

$$x^3 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} (x)_1 + \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} (x)_2 + \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} (x)_3 = 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2)$$

  Expand:

$$x + 3(x^2 - x) + (x^3 - 3x^2 + 2x) = x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3$$

  Verified: $x^3 = x^3$.

- **Lah Numbers Identity:**

$$x^{(3)} = \left\lfloor {3 \atop 1} \right\rfloor (x)_1 + \left\lfloor {3 \atop 2} \right\rfloor (x)_2 + \left\lfloor {3 \atop 3} \right\rfloor (x)_3 = 6x + 6 \cdot x(x-1) + 1 \cdot x(x-1)(x-2)$$

  Expand:

$$6x + 6(x^2 - x) + (x^3 - 3x^2 + 2x) = 6x + 6x^2 - 6x + x^3 - 3x^2 + 2x = x^3 + 3x^2 + 2x$$

  Verified: $x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

Let us denote the Stirling and Lah numbers as follows:

$$s(n,k) := (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{(signed Stirling numbers of the first kind),}$$

$$L(n,k) := (-1)^{n-k} \left\lfloor {n \atop k} \right\rfloor \quad \text{(signed Lah numbers),}$$

$$S(n,k) := \begin{Bmatrix} n \\ k \end{Bmatrix} \quad \text{(Stirling numbers of the second kind).}$$

Exercise.

Show the following identities:

(a) $x^n = \sum_{k=0}^{n} (-1)^{n-k} S(n,k) x^{(k)}$,

(b) $(x)_n = \sum_{k=0}^{n} s(n,k) x^k$,

(c) $(x)_n = \sum_{k=0}^{n} L(n,k) x^{(k)}$,

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial and $x^{(k)} = x(x+1)\cdots(x+k-1)$ is the rising factorial.

2.5 Lah Numbers and Stirling Number Identities

Theorem 2.9 (Lah Numbers as a Convolution of Stirling Numbers).
For integers $n \geq k \geq 0$, the Lah numbers satisfy:

$$\left\lfloor {n \atop k} \right\rfloor = \sum_{j=k}^{n} \left[{n \atop j} \right] \left\{ {j \atop k} \right\}.$$

2.6 Stirling Number Identities

Theorem 2.10 (Orthogonality of Stirling Numbers).
For integers $n \geq m \geq 0$, the unsigned Stirling numbers of the first kind $\left[{n \atop j} \right]$ and the Stirling numbers of the second kind $\left\{ {j \atop m} \right\}$ satisfy:

$$\sum_{j=m}^{n} \left[{n \atop j} \right] \left\{ {j \atop m} \right\} (-1)^{j-m} = \delta_{nm},$$
$$\sum_{j=m}^{n} \left\{ {n \atop j} \right\} \left[{j \atop m} \right] (-1)^{j-m} = \delta_{nm},$$

where δ_{nm} is the Kronecker delta (1 if $n = m$, 0 otherwise).

Exercise.
Prove Theorem 2.10.

Theorem 2.11 (Stirling Number Inversion).
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

(i) $a_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right] a_k$.

(ii) $b_n = \sum_{k=0}^{n} \left[{n \atop k} \right] a_k \Leftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} b_k$.

Exercise.
Prove Theorem 2.11.

Theorem 2.12 (Binomial Convolutions for Stirling Numbers).
The Stirling numbers of the first and second kinds satisfy the following convolution identities:

(i) **Horizontal convolution (First Kind):** For integers $n \geq 0$ and $k \geq 0$,
$$\left[{n+1 \atop k+1} \right] = \sum_{j=0}^{n} \left[{n \atop j} \right] \binom{j}{k}.$$

(ii) **Vertical convolution (Second Kind):** For integers $n \geq 0$ and $k \geq 0$,
$$\left\{ {n+1 \atop k+1} \right\} = \sum_{j=0}^{n} \binom{n}{j} \left\{ {j \atop k} \right\}.$$

Exercise.
Prove Theorem 2.12.

2.7 Generating Functions of Stirling numbers

Theorem 2.13 (Bivariate Generating Functions for Stirling Numbers).
For complex numbers t and u:

(i) **Unsigned Stirling numbers of the first kind**:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[{n \atop k}\right]\frac{t^k u^n}{n!} = (1-u)^{-t}.$$

(ii) **Signed Stirling numbers of the first kind**:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} s(n,k)\frac{t^k u^n}{n!} = (1+u)^{t}.$$

(iii) **Stirling numbers of the second kind**:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{{n \atop k}\right\}(t)_k\frac{u^n}{n!} = e^{ut}.$$

Theorem 2.14 (Closed Formula for Stirling Numbers of the Second Kind)**.**
The Stirling number $\left\{{n \atop k}\right\}$ is given by:

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

2.8 Exponential Generating Functions for Fixed k

Theorem 2.15 (EGFs for Stirling Numbers)**.**
For integers $k \geq 0$ and complex u:

(i) **Unsigned Stirling numbers of the first kind** ($\left[{n \atop k}\right]$):

$$\sum_{n=k}^{\infty}\left[{n \atop k}\right]\frac{u^n}{n!} = \frac{(-\ln(1-u))^k}{k!}.$$

(ii) **Signed Stirling numbers of the first kind** ($s(n,k) = (-1)^{n-k}\left[{n \atop k}\right]$):

$$\sum_{n=k}^{\infty} s(n,k)\frac{u^n}{n!} = \frac{(\ln(1+u))^k}{k!}.$$

(iii) **Stirling numbers of the second kind** ($\left\{{n \atop k}\right\}$):

$$\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}\frac{u^n}{n!} = \frac{(e^u-1)^k}{k!}.$$

3. Bell Numbers

Definition 3.1.
The Bell numbers count the number of possible partitions of a set of n elements into non-empty subsets, denoted B_n.

Proposition 3.2.
The n-th Bell number is given by

$$B_n = \sum_{k=0}^{n}\left\{{n \atop k}\right\},$$

where $\left\{{n \atop k}\right\}$ denotes the Stirling number of the second kind.

Example.
For $n = 3$, consider the set $\{a, b, c\}$. The Bell number B_3 counts these 5 partitions:

1. One subset: $\big\{\{a,b,c\}\big\}$
2. Two subsets:
   - $\big\{\{a\},\{b,c\}\big\}$
   - $\big\{\{b\},\{a,c\}\big\}$
   - $\big\{\{c\},\{a,b\}\big\}$
3. Three subsets: $\big\{\{a\},\{b\},\{c\}\big\}$

Thus, $B_3 = 5$. Using the formula:

$$B_3 = \sum_{k=0}^{3} {3 \brace k} = {3 \brace 1} + {3 \brace 2} + {3 \brace 3} = 1 + 3 + 1 = 5.$$

Theorem 3.3 (Binomial Convolution for Bell Numbers)**.**
The Bell numbers satisfy:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Theorem 3.4 (Exponential Generating Function for Bell Numbers)**.**
The Bell numbers satisfy:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Theorem 3.5 (Dobiński's Formula)**.**
For the n-th Bell number B_n,

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

4. Symmetric polynomial and Stirling numbers

Theorem 4.1.
For integers $n \geq k \geq 1$, the following identities hold:
1. **Unsigned Stirling Numbers of the First Kind**:

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = \sum_{0 \leq i_1 < i_2 < \cdots < i_k < n} i_1 i_2 \cdots i_k,$$

where the sum runs over all strictly increasing sequences $i_1, i_2, \ldots, i_k$ in $\{0, 1, \ldots, n-1\}$.
2. **Stirling Numbers of the Second Kind**:

$${n \brace k} = \frac{1}{k!} \sum_{\substack{i_1 + i_2 + \cdots + i_k = n \\ i_j \geq 1}} \frac{n!}{i_1! i_2! \cdots i_k!},$$

where the sum is over all compositions of n into k positive integers i_j.
3. Additionally, the **Unsigned Stirling Numbers of the First Kind** satisfy:

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = n! \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{i_1 i_2 \cdots i_k}$$

Example.

1. For $n = 4$, $k = 2$:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \sum_{0 \le i_1 < i_2 < 4} i_1 i_2 = 0 \cdot 1 + 0 \cdot 2 + 0 \cdot 3 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11.$$

2. For $n = 4$, $k = 2$:

$$\begin{Bmatrix} 4 \\ 2 \end{Bmatrix} = \frac{1}{2!} \sum_{\substack{i_1 + i_2 = 4 \\ i_j \ge 1}} \frac{24}{i_1! i_2!} = \frac{1}{2} \left(\frac{24}{1!3!} + \frac{24}{2!2!} + \frac{24}{3!1!} \right) = \frac{1}{2}(4 + 6 + 4) = 7.$$

3. For $n = 4$, $k = 2$:

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = 4! \sum_{1 \le i_1 < i_2 \le 4} \frac{1}{i_1 i_2} = 24 \left(\frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} \right).$$

Sum: $\frac{35}{24}$. Multiply: $24 \cdot \frac{35}{24} = 35$.