Chapter 1: Solving Recurrence Relations

1. Introduction to Recurrence Relations

Definition 1.1 (Sequence).
A **sequence** is a function

$$a : X \to S,$$

where X is a subset of consecutive integers, typically $\mathbb{N}_0$ or $\mathbb{N}$. The value $a(i)$, denoted by a_i, is called the ith term of the sequence.

- A *finite sequence* has $X = \{i_1, i_2, \dots, i_n\}$ and is written as $(a_{i_1}, a_{i_2}, \dots, a_{i_n})$.
- An *infinite sequence* has $X = \mathbb{N}_0$ or $\mathbb{N}$ and is denoted

$$(a_n)_{n\in\mathbb{N}_0} = (a_0, a_1, a_2, \dots) \quad \text{or} \quad (a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots).$$

Definition 1.2 (Recursive Sequence).
A sequence $(a_n)_{n\in\mathbb{N}_0}$ is called a **recursive sequence of order** k if there exists a function $F : S^k \to S$ and initial values $a_0, a_1, \dots, a_{k-1} \in S$ such that

$$a_n = \begin{cases} a_0, a_1, \dots, a_{k-1} \text{ are given as initial values} & \text{if } 0 \le n < k, \\ F(a_{n-1}, a_{n-2}, \dots, a_{n-k}), & \text{if } n \ge k. \end{cases}$$

Definition 1.3 (Linear Recurrence Relation).
A recursive sequence is called a **linear recurrence relation of order** k if it can be written as:

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} + g(n) \quad \text{for } n \ge k,$$

where $c_1, c_2, \dots, c_k$ are coefficients (which may depend on n) with $c_k \ne 0$, and $g(n)$ is a given function of n.

- If $g(n) = 0$, the recurrence is *homogeneous*.
- If $g(n) \ne 0$, the recurrence is *nonhomogeneous*.

Examples (Recursive Sequences).
The following are fundamental examples of recursively defined sequences:

- **Sequence of Positive Integers:**

$$a_n = \begin{cases} 1, & \text{if } n = 1, \\ a_{n-1} + 1, & \text{if } n \ge 2. \end{cases}$$

- **Arithmetic Progression:** A sequence with initial term a_0 and common difference d:

$$a_n = \begin{cases} a_0, & \text{if } n = 0, \\ a_{n-1} + d, & \text{if } n \ge 1. \end{cases}$$

- **Geometric Progression:** A sequence with initial term a_0 and common ratio r:

$$a_n = \begin{cases} a_0, & \text{if } n = 0, \\ r \cdot a_{n-1}, & \text{if } n \ge 1. \end{cases}$$

- **Fibonacci Sequence:** Defined by

$$F_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-1} + F_{n-2}, & \text{if } n \ge 2. \end{cases}$$

- **Logistic Map:** This recurrence is nonlinear because it involves a product of terms.

$$a_n = r \cdot a_{n-1}(1 - a_{n-1}),$$

  where r is a constant.

Definition 1.4 (Closed Form).
A *closed form* for the term a_n is an expression $E(n)$ such that

$$a_n = E(n),$$

that can be evaluated in a finite number of standard operations (addition, multiplication, exponentiation, etc.) with no dependence on prior terms of the sequence.

Remark.
The *order* k indicates the number of preceding terms used to define a_n.

1.1 Closed Forms for Elementary Finite Sums

In this subsection, we list some common finite sums and show simple ways to derive their closed forms. A *closed form* is a formula that lets you compute the sum directly, without adding term by term.

Gauss's Sum of Positive Integers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. Let

$$S = 1 + 2 + \cdots + n.$$

Write the same sum in reverse order:

$$S = n + (n-1) + \cdots + 1.$$

Now add the two equal expressions term by term:

$$2S = (1+n) + (2+(n-1)) + \cdots + (n+1).$$

Each pair equals $n+1$, and there are n pairs, so

$$2S = n(n+1).$$

Divide both sides by 2 to get

$$S = \frac{n(n+1)}{2}.$$

□

Sum of the First n Odd Numbers

$$\sum_{k=1}^{n} (2k-1) = n^2.$$

Proof. We prove this by induction on n.
Base case ($n = 1$).

$$\sum_{k=1}^{1} (2k-1) = 1 = 1^2.$$

Inductive step. Assume the formula is true for some n, that is,

$$\sum_{k=1}^{n} (2k-1) = n^2.$$

Then for $n+1$,

$$\sum_{k=1}^{n+1} (2k-1) = \left(\sum_{k=1}^{n} (2k-1)\right) + (2(n+1)-1) = n^2 + (2n+1) = (n+1)^2.$$

So the formula holds for $n+1$. Therefore, it holds for all $n \geq 1$. □

Sum of the First n Even Numbers

$$\sum_{k=1}^{n} 2k = n(n+1).$$

Proof. Factor out 2:

$$\sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k.$$

Using Gauss's formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, we get

$$2\sum_{k=1}^{n} k = 2 \cdot \frac{n(n+1)}{2} = n(n+1).$$

□

Arithmetic Progression Sum

For the arithmetic sequence

$$a_1,\ a_1+d,\ a_1+2d,\ \dots,\ a_1+(n-1)d,$$

the sum of the first n terms is

$$S_n = \frac{n}{2}\Big(2a_1 + (n-1)d\Big).$$

Proof. Write the sum in two ways:

$$S_n = a_1 + (a_1+d) + \cdots + \Big(a_1+(n-1)d\Big),$$

$$S_n = \Big(a_1+(n-1)d\Big) + \Big(a_1+(n-2)d\Big) + \cdots + a_1.$$

Add term by term. Each pair equals $2a_1+(n-1)d$, and there are n pairs:

$$2S_n = n\Big(2a_1+(n-1)d\Big).$$

Divide by 2:

$$S_n = \frac{n}{2}\Big(2a_1 + (n-1)d\Big).$$

□

Geometric Progression Sum

For the geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ \dots,\ a_1r^{n-1} \quad (r \neq 1),$$

the sum of the first n terms is

$$S_n = a_1\frac{1-r^n}{1-r}.$$

Proof. Let

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}.$$

Multiply both sides by r:

$$rS_n = a_1r + a_1r^2 + \cdots + a_1r^n.$$

Subtract the second equation from the first:

$$S_n - rS_n = a_1 - a_1r^n.$$

Factor:

$$(1-r)S_n = a_1(1-r^n).$$

Since $r \neq 1$, divide by $1-r$:

$$S_n = a_1\frac{1-r^n}{1-r}.$$

□

Sum of Powers (Examples)

For $p \in \mathbb{N}$, the sum $\sum_{k=1}^{n} k^p$ has a closed form (we will not derive the general formula here). Two important examples are:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. These formulas can be proven by induction.

□

Lemma 1.5.
For any integer $n \geq 1$ and any real numbers x and y,

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-1-k} = (x-y)\sum_{k=1}^{n} x^{k-1} y^{n-k}.$$

Proof. Let

$$S = \sum_{k=0}^{n-1} x^k y^{n-1-k} = y^{n-1} + xy^{n-2} + x^2 y^{n-3} + \cdots + x^{n-1}.$$

Multiply S by x and by y:

$$xS = xy^{n-1} + x^2 y^{n-2} + \cdots + x^n, \qquad yS = y^n + xy^{n-1} + \cdots + x^{n-1}y.$$

Now subtract:

$$xS - yS = (x^n - y^n).$$

Factor the left-hand side:

$$(x-y)S = x^n - y^n.$$

So

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

The second form is the same identity with the index shift $k \mapsto k-1$. □

2. First-Order Recurrence Relations

2.1 General first-order linear recurrence

We consider the first-order linear recurrence relation

$$U_n = \begin{cases} U_0, & n = 0, \\ c_n U_{n-1} + g(n), & n \geq 1, \end{cases}$$

where $(c_n)_{n\geq 1}$ is a given sequence of coefficients and $g(n)$ is a given function.

Idea (repeated substitution). We replace U_{n-1} using the recurrence, then replace U_{n-2}, and so on. A pattern appears:

- products of the coefficients c_j,
- a weighted sum of the terms $g(i)$.

Theorem 2.1.
For every $n \geq 1$, the closed form of U_n is

$$U_n = \left(\prod_{j=1}^{n} c_j\right) U_0 + \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} c_j\right) g(i).$$

Proof. We expand the recurrence step by step and keep track of the pattern.

Step 1: expand a few times.

$$\begin{aligned} U_n &= c_n U_{n-1} + g(n), \\ &= c_n(c_{n-1}U_{n-2} + g(n-1)) + g(n) \\ &= c_n c_{n-1} U_{n-2} + c_n g(n-1) + g(n), \\ &= c_n c_{n-1}(c_{n-2}U_{n-3} + g(n-2)) + c_n g(n-1) + g(n) \\ &= c_n c_{n-1} c_{n-2} U_{n-3} + c_n c_{n-1} g(n-2) + c_n g(n-1) + g(n). \end{aligned}$$

Step 2: state the general pattern. After expanding until U_0, we obtain

$$U_n = \left(\prod_{j=1}^{n} c_j\right) U_0 + \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} c_j\right) g(i).$$

(Each time we substitute once more, the coefficient in front of U_{n-t} becomes $\prod_{j=n-t+1}^{n} c_j$, and a new term involving $g(n-t+1)$ appears with the correct product.)

Step 3: simplify the last term. When $i = n$, the product $\prod_{j=n+1}^{n} c_j$ is an empty product and equals 1, so the last summand is exactly $g(n)$. This matches the formula above. ☐

2.2 Constant coefficient case and geometric sums

The most common situation is when the coefficient is constant: $c_n = a$.

Corollary 2.2.
If

$$U_n = \begin{cases} U_0, & n = 0, \\ aU_{n-1} + g(n), & n \geq 1, \end{cases}$$

then for $n \geq 1$,

$$U_n = a^n U_0 + \sum_{i=1}^{n} a^{n-i} g(i).$$

Proof. Apply Theorem 2.1 with $c_j = a$ for all j. Then $\prod_{j=1}^{n} c_j = a^n$ and $\prod_{j=i+1}^{n} c_j = a^{n-i}$. ☐

A standard model: $U_n = aU_{n-1} + b$

Corollary 2.3.
The recurrence

$$U_n = \begin{cases} U_0, & n = 0, \\ aU_{n-1} + b, & n \geq 1, \end{cases}$$

has the closed form

$$U_n = \begin{cases} a^n U_0 + b\dfrac{a^n - 1}{a - 1}, & a \neq 1, \\ U_0 + nb, & a = 1. \end{cases}$$

Proof. From Corollary 2.2,

$$U_n = a^n U_0 + \sum_{i=1}^{n} a^{n-i} b = a^n U_0 + b \sum_{i=1}^{n} a^{n-i} = a^n U_0 + b \sum_{t=0}^{n-1} a^t.$$

If $a \neq 1$, we use the geometric sum $\sum_{t=0}^{n-1} a^t = \dfrac{a^n - 1}{a - 1}$. If $a = 1$, then $\sum_{t=0}^{n-1} 1 = n$, so $U_n = U_0 + nb$. ☐

Exercise.
Solve the recurrence

$$U_n = \frac{2}{3} U_{n-1} + 1, \qquad U_0 = 0.$$

Solution.
Here $a = \frac{2}{3}$ and $b = 1$. By Corollary 2.3 (with $U_0 = 0$),

$$U_n = 1 \cdot \frac{\left(\frac{2}{3}\right)^n - 1}{\frac{2}{3} - 1} = 3\left(1 - \left(\frac{2}{3}\right)^n\right).$$

Remark (How to check your answer)**.**
To verify a closed form, you should:

- check the initial value,
- substitute the formula into $U_n = aU_{n-1} + b$ and simplify,
- and prove it by induction on n.

2.3 Example: Tower of Hanoi

The Tower of Hanoi is a puzzle with three pegs (A, B, C) and n disks of different sizes. Initially, all disks are on peg A (largest at the bottom). The goal is to move all disks to peg C under these rules:

1. move only one disk at a time,

2. never place a larger disk on a smaller disk,
3. move disks only between the pegs.

Let T_n be the minimum number of moves needed with n disks. For small n:

$$T_1 = 1, \qquad T_2 = 3, \qquad T_3 = 7.$$

Recurrence. To move n disks from A to C:

1. move the top $n-1$ disks from A to B: T_{n-1} moves,
2. move the largest disk from A to C: 1 move,
3. move the $n-1$ disks from B to C: T_{n-1} moves.

So

$$T_n = 2T_{n-1} + 1, \qquad T_0 = 0.$$

Closed form. This is the case $a = 2$, $b = 1$, $T_0 = 0$. Using Corollary 2.3,

$$T_n = 2^n \cdot 0 + \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

General Method (Finding a Recurrence)

1. **Define the quantity.** Let U_n be the number of valid solutions for a problem of size n.
2. **Split into cases.** Partition all valid solutions into **disjoint cases** using a simple structural choice. The cases must **cover all solutions** with **no double counting**.
3. **Reduce to smaller sizes.** In each case, show how a size-n solution determines a valid solution of smaller size (such as $n-1$, $n-2$, $\ldots$). Express the number of solutions in that case using $U_{n-1}, U_{n-2}, \ldots$.
4. **Combine the counts.** Add the contributions of all cases. This gives the **recurrence relation** for U_n.
5. **Give initial conditions.** Compute U_0, U_1 (and sometimes U_2) directly from the definition. This makes the recurrence **well-defined**.

Easy Example: Counting Permutations

- **Define:** Let U_n be the number of ways to arrange n distinct books on a shelf (that is, the number of permutations of n objects).
- **Split into disjoint cases (by the first position):** In every arrangement, exactly one book is placed in the first position. There are n choices for this first book, so we obtain n **disjoint cases**, and these cases **cover all solutions** with **no double counting**.
- **Reduce to a smaller problem:** Once the first book is chosen and fixed, the remaining $n-1$ books can be arranged in U_{n-1} ways.
- **Combine the counts:** Each of the n cases contributes U_{n-1} arrangements, so the total number of arrangements is
$$U_n = n \cdot U_{n-1}.$$
- **Initial condition:** With 0 books, there is exactly one arrangement (the empty arrangement), so
$$U_0 = 1.$$

  This makes the sequence well-defined and yields the factorial numbers $U_n = n!$.

3. Second-Order Recurrence Relations

A second-order recurrence relation expresses each term in terms of the two preceding terms. It is given by:

$$U_n = aU_{n-1} + bU_{n-2} + g(n), \quad \text{for } n \geq 2,$$

where a and b are constants, and $g(n)$ is a given function. The initial values U_0 and U_1 are given.

3.1 Characteristic Equation Method

In the previous section, we discussed how to solve first-order recurrence relations. The approach for solving second-order recurrence relations involves transforming them into first-order recurrence relations and then solving them accordingly.

To achieve this, we introduce the following change of variable:

$$V_n = U_n - \alpha U_{n-1}, \tag{1}$$

where V_n satisfies a first-order recurrence relation:

$$V_n = \beta V_{n-1} + g(n). \tag{2}$$

Substituting (1) into (2) , we obtain:

$$U_n - \alpha U_{n-1} = \beta(U_{n-1} - \alpha U_{n-2}) + g(n).$$

Rearranging, we get:

$$U_n = (\alpha + \beta)U_{n-1} - \alpha\beta U_{n-2} + g(n).$$

Comparing this with the standard second-order recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2} + g(n),$$

we conclude that the parameters must satisfy the system:

$$\begin{cases} \alpha + \beta = a, \\ \alpha\beta = -b. \end{cases} \tag{3}$$

Lemma 3.1 (Vieta's Formulas for Quadratic Equations).
Let x and y be two real (or complex) numbers. If

$$x + y = a \quad \text{and} \quad xy = b,$$

then x and y are the roots of the quadratic equation

$$t^2 - at + b = 0. \tag{4}$$

Proof. Since x and y are two numbers, we can form the polynomial

$$(t - x)(t - y).$$

Expanding, we obtain

$$(t - x)(t - y) = t^2 - (x + y)t + xy.$$

Using the assumptions $x + y = a$ and $xy = b$, this becomes

$$t^2 - at + b.$$

Therefore,

$$(t - x)(t - y) = t^2 - at + b,$$

which shows that x and y are the roots of the equation $t^2 - at + b = 0$. □

Thus, solving (33) reduces to finding the roots of the characteristic equation:

$$(t - \alpha)(t - \beta) = t^2 - (\alpha + \beta)t + \alpha\beta = t^2 - at - b = 0. \tag{5}$$

Once we determine α and β, we return to (2), which gives the first-order recurrence:

$$V_n = \beta V_{n-1} + g(n).$$

Using standard techniques, we solve this recurrence to find its general solution, denoted as $h(n)$. Substituting back into (1), we obtain:

$$U_n = \alpha U_{n-1} + h(n).$$

Finally, solving this first-order recurrence provides the explicit expression for U_n.

Remark.
As with all recurrence relations, any closed-form solution obtained must be verified by mathematical induction.

3.1.1 Example: Fibonacci Sequence

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2, \tag{6}$$

with initial conditions:

$$F_0 = 0, \quad F_1 = 1. \tag{7}$$

To transform this second-order recurrence into a first-order recurrence, we introduce the change of variable:

$$V_n = F_n - \alpha F_{n-1}. \tag{8}$$

where V_n satisfies a first-order recurrence relation:

$$V_n = \beta V_{n-1}. \tag{9}$$

Substituting (8) into (9), we obtain:

$$F_n - \alpha F_{n-1} = \beta(F_{n-1} - \alpha F_{n-2}),$$

which simplifies to:

$$F_n = (\alpha + \beta)F_{n-1} - \alpha\beta F_{n-2}.$$

Comparing with (6), we conclude that the parameters satisfy:

$$\begin{cases} \alpha + \beta = 1, \\ \alpha\beta = -1. \end{cases} \tag{10}$$

Solving for α and β, we derive the characteristic polynomial:

$$(t - \alpha)(t - \beta) = t^2 - (\alpha + \beta)t + \alpha\beta = t^2 - t - 1 = 0. \tag{11}$$

Applying the quadratic formula:

$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.$$

Thus, the characteristic roots are:

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}. \tag{12}$$

Remark.
The quantity

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

is called the **Golden Ratio**. It plays an important role in mathematics, nature, and various applications in art and architecture. The second root,

$$\bar{\varphi} = \frac{1 - \sqrt{5}}{2},$$

is often referred to as the *conjugate* of the Golden Ratio.

Remark.
The choice of assigning α and β as the first and second solutions, or vice versa, does not affect the final result. This is because the solution structure remains the same, and any change in ordering is absorbed by the constants determined by initial conditions.

Since we obtained α and β, we now solve the first-order recurrence:

$$V_n = \beta V_{n-1}.$$

Using standard techniques, we solve this recurrence to find its general solution:

$$V_n = \beta^{n-1} V_1. \tag{13}$$

From (8), we set $V_1 = F_1 = 1$, so:

$$V_n = \beta^{n-1}, \quad \text{where } \beta = \frac{1 - \sqrt{5}}{2}.$$

Substituting back into (8), we obtain:

$$\begin{aligned} F_n &= \alpha F_{n-1} + \beta^{n-1}, \\ \alpha F_{n-1} &= \alpha^2 F_{n-2} + \alpha\beta^{n-2}, \\ \alpha^2 F_{n-2} &= \alpha^3 F_{n-3} + \alpha^2\beta^{n-3}, \\ \vdots &= \vdots + \vdots, \\ \alpha^{n-2} F_2 &= \alpha^{n-1} F_1 + \alpha^{n-2}\beta^1, \\ \alpha^{n-1} F_1 &= \alpha^n F_0 + \alpha^{n-1}. \end{aligned}$$

Summing all the equations, we obtain the explicit formula for the Fibonacci sequence (Binet's formula):

$$\begin{aligned} F_n &= \sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right). \end{aligned}$$

Proof by Induction We now prove by induction that:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \bar{\varphi}^n \right), \tag{14}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ is its conjugate.

Base Cases: For $n = 0$:

$$F_0 = \frac{1}{\sqrt{5}} \left(\varphi^0 - \bar{\varphi}^0 \right) = \frac{1}{\sqrt{5}}(1-1) = 0.$$

For $n = 1$:

$$F_1 = \frac{1}{\sqrt{5}} \left(\varphi^1 - \bar{\varphi}^1 \right) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right).$$

Since $\varphi - \bar{\varphi} = \sqrt{5}$, we obtain:

$$F_1 = \frac{\sqrt{5}}{\sqrt{5}} = 1.$$

Inductive Step: Assume that the formula holds for n and $n-1$:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \bar{\varphi}^n \right).$$

Using the Fibonacci recurrence and properties of φ, we conclude:

$$F_{n+1} = \frac{1}{\sqrt{5}} \left(\varphi^{n+1} - \bar{\varphi}^{n+1} \right).$$

Thus, Binet's formula is proven by induction.

Exercise.

The approach used to solve second-order recurrence relations can be extended to third-order recurrence relations. Consider the recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2} + cU_{n-3},$$

where the initial values U_0, U_1, and U_2 are given.

- Propose a systematic method to solve this recurrence relation.
- Derive the associated characteristic polynomial.

3.2 Generating Functions for a Second-Order Homogeneous Recurrence

In this subsection, we present the generating function method for solving a second-order homogeneous linear recurrence relation.

We consider the recurrence

$$U_n = aU_{n-1} + bU_{n-2}, \qquad n \geq 2, \tag{15}$$

where a, b are constants and the initial values U_0 and U_1 are given.

Our objective is to determine an explicit formula for U_n.

Step 1: Definition of the Generating Function

Definition 3.2.
The ordinary generating function of the sequence $(U_n)_{n\geq 0}$ is

$$G(x) = \sum_{n=0}^{\infty} U_n x^n. \tag{16}$$

Explicitly,

$$G(x) = U_0 + U_1 x + U_2 x^2 + U_3 x^3 + \cdots .$$

Step 2: Index Shifting Formulas

In order to use recurrence (15), we multiply both sides by x^n and sum for $n \geq 2$. We compute each term separately.

Lemma 3.3.
Let $G(x)$ be defined by (16). Then:

$$\sum_{n=2}^{\infty} U_n x^n = G(x) - U_0 - U_1 x, \tag{17}$$

$$\sum_{n=2}^{\infty} U_{n-1} x^n = x\Big(G(x) - U_0\Big), \tag{18}$$

$$\sum_{n=2}^{\infty} U_{n-2} x^n = x^2 G(x). \tag{19}$$

Proof. We prove (18); the others are similar.

$$\sum_{n=2}^{\infty} U_{n-1} x^n = x \sum_{n=2}^{\infty} U_{n-1} x^{n-1}.$$

Set $k = n - 1$. Then $k \geq 1$, and

$$x \sum_{k=1}^{\infty} U_k x^k = x\Big(G(x) - U_0\Big).$$

□

Step 3: Determination of the Generating Function

Theorem 3.4.
If (U_n) satisfies (15), then its generating function is

$$G(x) = \frac{U_0 + (U_1 - aU_0)x}{1 - ax - bx^2}. \tag{20}$$

Proof. Multiply (15) by x^n and sum for $n \geq 2$:

$$\sum_{n=2}^{\infty} U_n x^n = a \sum_{n=2}^{\infty} U_{n-1} x^n + b \sum_{n=2}^{\infty} U_{n-2} x^n.$$

Using Lemma 3.3, we obtain

$$G(x) - U_0 - U_1 x = ax\Big(G(x) - U_0\Big) + bx^2 G(x).$$

Collecting terms in $G(x)$ gives

$$(1 - ax - bx^2)G(x) = U_0 + (U_1 - aU_0)x,$$

which yields (20). □

Step 4: Factorization of the Denominator

We know that α and β are the solutions of the characteristic equation

$$T^2 - aT - b = 0. \tag{21}$$

By Vieta's formulas, we can write:

$$T^2 - aT - b = (T - \alpha)(T - \beta).$$

If we put $T = \frac{1}{x}$, the equation becomes:

$$\frac{1}{x^2} - \frac{a}{x} - b = \left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right).$$

Multiplying both sides by x^2, we obtain the factorization of the denominator:

$$1 - ax - bx^2 = (1 - \alpha x)(1 - \beta x). \tag{22}$$

Step 5: Explicit Formula for U_n

Case 1: Distinct roots ($\alpha \neq \beta$). Using the factorization (22), we can rewrite the generating function $G(x)$:

$$G(x) = \frac{U_0 + (U_1 - aU_0)x}{1 - ax - bx^2} = \frac{U_0 + (U_1 - aU_0)x}{(1 - \alpha x)(1 - \beta x)}.$$

The decomposition of $G(x)$ is as follows:

$$G(x) = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x} = A \sum_{n \geq 0} \alpha^n x^n + B \sum_{n \geq 0} \beta^n x^n.$$

From this series expansion, we can identify the general term U_n:

$$U_n = A\alpha^n + B\beta^n.$$

The constants A and B satisfy the following system of equations (by evaluating at $n = 0$ and $n = 1$):

$$\begin{cases} U_0 = A + B \\ U_1 = \alpha A + \beta B \end{cases}$$

Solving this system to isolate A, we get:

$$U_1 - \beta U_0 = (\alpha - \beta)A \implies A = \frac{U_1 - \beta U_0}{\alpha - \beta}.$$

Similarly, solving the system to isolate B, we get:

$$U_1 - \alpha U_0 = (\beta - \alpha)B \implies B = -\frac{U_1 - \alpha U_0}{\alpha - \beta}.$$

Substituting A and B back into our expression for U_n, and factoring out $\frac{1}{\alpha - \beta}$, we get the final explicit formula:

$$U_n = \frac{1}{\alpha - \beta}\Big[(U_1 - \beta U_0)\alpha^n - (U_1 - \alpha U_0)\beta^n\Big]. \tag{23}$$

Case 2: Double root ($\alpha = \beta$). Assume that the characteristic equation (21) has a double root α. Then

$$1 - ax - bx^2 = (1 - \alpha x)^2.$$

If a denominator contains a repeated linear factor $(1 - \alpha x)^2$, then the partial fraction decomposition of $G(x)$ must be of the form:

$$G(x) = \frac{U_0 + (U_1 - aU_0)x}{(1 - \alpha x)^2} = \frac{A}{1 - \alpha x} + \frac{B}{(1 - \alpha x)^2}. \tag{24}$$

Using the known power series expansions:

$$\frac{1}{1 - \alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n, \qquad \frac{1}{(1 - \alpha x)^2} = \sum_{n=0}^{\infty} (n + 1)\alpha^n x^n,$$

we can rewrite the generating function as:

$$G(x) = A \sum_{n=0}^{\infty} \alpha^n x^n + B \sum_{n=0}^{\infty} (n + 1)\alpha^n x^n.$$

From this series expansion, we can identify the general term U_n by factoring out α^n:

$$U_n = \Big[A + B(n+1)\Big]\alpha^n = \Big[(A+B) + Bn\Big]\alpha^n.$$

To simplify, we define new constants $C_1 = A + B$ and $C_2 = B$, so that the general term takes the standard form:

$$U_n = (C_1 + C_2 n)\alpha^n. \tag{25}$$

The constants C_1 and C_2 satisfy the following system of equations (by evaluating at $n = 0$ and $n = 1$):

$$\begin{cases} U_0 = C_1 \\ U_1 = (C_1 + C_2)\alpha \end{cases}$$

From the first equation, we immediately get:

$$C_1 = U_0.$$

Substituting C_1 into the second equation and dividing by α to isolate C_2, we get:

$$C_1 + C_2 = \frac{U_1}{\alpha} \implies C_2 = \frac{U_1}{\alpha} - U_0.$$

This completes the double root case.

Formulas (23) and (25) give the explicit solution of recurrence (15). The explicit formula for the sequence (U_n) can be summarized compactly as follows:

$$U_n = \begin{cases} \alpha^n U_0 + \dfrac{\alpha^n - \beta^n}{\alpha - \beta}\left(U_1 - \alpha U_0\right), & \text{if } \alpha \neq \beta, \\ \alpha^n U_0 + n\alpha^{n-1}\left(U_1 - \alpha U_0\right), & \text{if } \alpha = \beta. \end{cases} \tag{26}$$

3.3 Some properties of Second-Order Recurrence Relation

3.3.1 Binet's Formula

Binet's formula is simply the closed-form expression for any second-order recurrence sequence. For the standard Fibonacci sequence, it is given by:

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \bar{\varphi}^n\right), \tag{27}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \quad \bar{\varphi} = \frac{1-\sqrt{5}}{2}$$

are the two roots of the characteristic equation $t^2 - t - 1 = 0$.

Binet's Formula for Homogeneous Second-Order Linear Recurrence Sequence:

We now extend the same method to solve the general homogeneous second-order linear recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}, \quad \text{for } n \geq 2, \tag{28}$$

where the initial values U_0 and U_1 are given.

We consider the general second-order linear recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}, \quad \text{for } n \geq 2, \tag{29}$$

with initial conditions:

$$U_0, \quad U_1 \quad \text{given}. \tag{30}$$

Note that in particular, if the initial conditions are $U_0 = 0$ and $U_1 = 1$, this specific sequence is sometimes referred to as the generalized Fibonacci sequence.

To simplify this second-order recurrence into a first-order recurrence, we introduce the change of variable:

$$V_n = U_n - \alpha U_{n-1}, \tag{31}$$

where V_n satisfies the first-order recurrence:

$$V_n = \beta V_{n-1}. \tag{32}$$

Substituting (31) into (32), we have:

$$U_n - \alpha U_{n-1} = \beta(U_{n-1} - \alpha U_{n-2}),$$

which simplifies to:

$$U_n = (\alpha + \beta)U_{n-1} - \alpha\beta U_{n-2}.$$

Comparing with (29), we derive the system of equations:

$$\begin{cases} \alpha + \beta = a, \\ \alpha\beta = -b. \end{cases} \tag{33}$$

The characteristic polynomial for the recurrence is:

$$t^2 - at - b = 0. \tag{34}$$

Applying the quadratic formula, we find its characteristic roots:

$$t = \frac{a \pm \sqrt{a^2 + 4b}}{2}.$$

Thus, the characteristic roots are:

$$\beta = \frac{a + \sqrt{a^2 + 4b}}{2}, \quad \alpha = \frac{a - \sqrt{a^2 + 4b}}{2}. \tag{35}$$

Since we obtained α and β, we now solve the first-order recurrence:

$$V_n = \beta V_{n-1}.$$

Solving this recurrence explicitly:

$$V_n = \beta^{n-1} V_1, \tag{36}$$

where $V_1 = U_1 - \alpha U_0$.

Substituting back into (31), we obtain:

$$\begin{aligned} U_n &= \alpha U_{n-1} + \beta^{n-1} V_1, \\ \alpha U_{n-1} &= \alpha^2 U_{n-2} + \alpha\beta^{n-2} V_1, \\ \alpha^2 U_{n-2} &= \alpha^3 U_{n-3} + \alpha^2\beta^{n-3} V_1, \\ \vdots &= \vdots + \vdots, \\ \alpha^{n-2} U_2 &= \alpha^{n-1} U_1 + \alpha^{n-2}\beta V_1, \\ \alpha^{n-1} U_1 &= \alpha^n U_0 + \alpha^{n-1} V_1. \end{aligned}$$

Summing all equations, we derive the explicit formula:

$$U_n = \alpha^n U_0 + \sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i} V_1$$

Finally, substituting $V_1 = U_1 - \alpha U_0$, we get the explicit form of the solution:

$$U_n = \begin{cases} \alpha^n U_0 + \dfrac{\alpha^n - \beta^n}{\alpha - \beta} \left(U_1 - \alpha U_0 \right), & \text{if } \alpha \neq \beta. \\ \alpha^n U_0 + n\alpha^{n-1} \left(U_1 - \alpha U_0 \right), & \text{if } \alpha = \beta. \end{cases} \tag{37}$$

where constants $\beta = \frac{a+\sqrt{a^2+4b}}{2}$, and $\alpha = \frac{a-\sqrt{a^2+4b}}{2}$.

In particular, if $U_0 = 0$ and $U_1 = 1$, we get the Binet formula of the generalized Fibonacci sequence, given by:

$$U_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta}, & \text{if } \alpha \neq \beta. \\ n\alpha^{n-1}, & \text{if } \alpha = \beta. \end{cases} \tag{38}$$

3.3.2 Cassini's Identity: Homogeneous Second-Order Linear Recurrence

Consider the homogeneous second-order linear recurrence relation defined by:

$$U_n = aU_{n-1} + bU_{n-2}, \quad U_0, U_1 \text{ given}.$$

Theorem 3.5 (Cassini's Identity for Homogeneous Second-Order Linear Recurrences)**.** Then, the generalized Cassini identity holds:

$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2 U_0 - U_1^2).$$

Proof. **Base Case:** For $n = 1$, we verify the identity:

$$U_2U_0 - U_1^2 = (-b)^{1-1}(U_2U_0 - U_1^2),$$

which holds trivially.

Inductive Step: Assume the identity holds for $n = k \geq 1$:

$$U_{k+1}U_{k-1} - U_k^2 = (-b)^{k-1}(U_2U_0 - U_1^2).$$

We need to prove the identity for $n = k + 1$:

$$U_{k+2}U_k - U_{k+1}^2 = (-b)^k(U_2U_0 - U_1^2).$$

Using the recurrence relation $U_{k+2} = aU_{k+1} + bU_k$, we have:

$$U_{k+2}U_k - U_{k+1}^2 = (aU_{k+1} + bU_k)U_k - U_{k+1}^2.$$

Expanding and rearranging terms gives:

$$= aU_{k+1}U_k + bU_k^2 - U_{k+1}^2.$$

Since $U_{k+1} = aU_k + bU_{k-1}$, substitute and simplify:

$$= aU_kU_{k+1} + bU_k^2 - (aU_k + bU_{k-1})U_{k+1}.$$

Simplifying carefully, we get:

$$= b(U_k^2 - U_{k+1}U_{k-1}).$$

Using the induction hypothesis, we have:

$$= b(-(-b)^{k-1}(U_2U_0 - U_1^2)) = (-b)^k(U_2U_0 - U_1^2),$$

thus confirming the identity for $n = k + 1$. By induction, the generalized Cassini identity holds for all $n \geq 1$. □

3.3.3 Matrix Representations

The Ring of 2×2 Matrices

Let $\mathbb{K}$ be a field (such as $\mathbb{R}$ or $\mathbb{C}$) or, more generally, a commutative ring. A 2×2 **matrix** over $\mathbb{K}$ is an array of four elements from $\mathbb{K}$, arranged in two rows and two columns:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where } a, b, c, d \in \mathbb{K}.$$

The set of all 2×2 matrices over $\mathbb{K}$ is denoted by $\mathcal{M}_2(\mathbb{K})$.

We define two operations on $\mathcal{M}_2(\mathbb{K})$:

- **Addition:** We add matrices component by component.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

- **Multiplication:** We multiply matrices using the "row-by-column" rule.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Under these operations, $(\mathcal{M}_2(\mathbb{K}), +, \times)$ forms a **ring with identity**. The multiplicative identity is the matrix:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which satisfies $AI_2 = I_2A = A$ for any matrix $A \in \mathcal{M}_2(\mathbb{K})$.

Note that matrix multiplication is generally **not commutative** ($AB \neq BA$). Therefore, if you multiply a matrix equation $AX = B$ by another matrix C, you must multiply on the exact same side:

$$CAX = CB \quad \text{(left multiplication)} \quad \text{or} \quad AXC = BC \quad \text{(right multiplication)}.$$

Scalar Multiplication: We can multiply a matrix by a scalar $\lambda \in \mathbb{K}$ by multiplying every element by λ:

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

Determinant and Invertibility

The **determinant** of a 2×2 matrix is a value in $\mathbb{K}$ given by:

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

A key property of the determinant is that it preserves multiplication: $\det(AB) = \det(A)\det(B)$.

A matrix A is **invertible** if and only if $\det(A) \neq 0$ (more precisely, if $\det(A)$ is an invertible element in the ring $\mathbb{K}$).

If A is invertible, its inverse is the matrix A^{-1} given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This inverse satisfies the property $AA^{-1} = A^{-1}A = I_2$.

Matrix Powers

For any integer $n \geq 0$, the n-th power of a square matrix A is defined by repeated multiplication:

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}, \quad \text{where } A^0 = I_2.$$

Theorem 3.6.
Let U_n be a sequence defined by the homogeneous recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2},$$

where U_0 and U_1 are given initial conditions, and a, b are real or complex numbers ($a, b \in \mathbb{R}$ or $\mathbb{C}$). Then, the sequence U_n satisfies the following matrix recurrence relation:

$$\begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} U_1 \\ U_0 \end{pmatrix}.$$

Proof. To verify the result, we compute the right-hand side:

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} = \begin{pmatrix} aU_n + bU_{n-1} \\ U_n \end{pmatrix}.$$

By the recurrence relation, we have $U_{n+1} = aU_n + bU_{n-1}$, which matches the first entry of the resulting matrix. The second entry is simply U_n, as expected. This confirms the matrix representation of the recurrence relation. □

If we introduce the vector X_n and the associated matrix A:

$$X_n = \begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$

the recurrence relation from the theorem can be rewritten compactly as a first-order matrix equation:

$$X_{n+1} = AX_n.$$

By iterating this relation, we can express any vector in terms of the initial vector X_1:

$$X_{n+1} = A^n X_1.$$

This leads to a natural mathematical question: **Can we find an explicit formula for the n-th power of the matrix A?**

Theorem 3.7.
Let (U_n) be a sequence satisfying the recurrence relation $U_n = aU_{n-1} + bU_{n-2}$, and let $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$.
For any integer $n \geq 1$, we have the matrix identity:

$$A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Equivalently, assuming the initial condition matrix is invertible, the $(n-1)$-th power of A is given by:

$$A^{n-1} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

Proof. We proceed by mathematical induction on n, proving the first (equivalent) form of the equation to simplify our calculations.

Base Case ($n = 1$): For $n = 1$, the left-hand side involves A^0, which is the identity matrix I_2:

$$I_2 \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}.$$

The right-hand side for $n = 1$ is exactly the same matrix. Thus, the base case holds.

Inductive Step: Assume that the formula holds for a certain integer $n \geq 1$:

$$A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

We want to show that it also holds for $n + 1$; that is, we need to prove:

$$A^{n} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_{n} \end{pmatrix}.$$

Starting from our inductive hypothesis, we multiply both sides on the left by the matrix A:

$$A \left[A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} \right] = A \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

This simplifies to:

$$A^{n} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Now, we perform the matrix multiplication on the right-hand side:

$$A^{n} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} aU_{n+1} + bU_n & aU_n + bU_{n-1} \\ U_{n+1} & U_n \end{pmatrix}.$$

By the definition of our sequence, we know that $aU_{n+1} + bU_n = U_{n+2}$ and $aU_n + bU_{n-1} = U_{n+1}$. Substituting these relations into the top row of our matrix yields:

$$A^{n} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_{n} \end{pmatrix}.$$

This is exactly what we needed to show. By the principle of mathematical induction, the formula holds for all integers $n \geq 1$. □

Cassini's Identity via Determinants

The matrix formulation we just established provides an incredibly elegant and direct proof for **Cassini's identity**.

Corollary 3.8 (Cassini's Identity)**.**
For a second-order linear recurrence sequence defined by $U_n = aU_{n-1} + bU_{n-2}$, the following relation holds for all $n \geq 1$:

$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2U_0 - U_1^2).$$

Proof. From the previous theorem, we have the matrix identity:

$$A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

We take the determinant of both sides. Recall that for any two square matrices X and Y, the determinant of their product is the product of their determinants, meaning $\det(XY) = \det(X)\det(Y)$. Applying this property yields:

$$\det\left(A^{n-1}\right) \det \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \det \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

First, we calculate the determinant of the companion matrix A:

$$\det(A) = \det\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = (a)(0) - (b)(1) = -b.$$

Using the property that $\det\left(A^k\right) = (\det A)^k$, we have $\det(A^{n-1}) = (-b)^{n-1}$.
Next, we calculate the determinants of the sequence matrices by applying the standard formula $ad - bc$:

$$\det\begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} = U_{n+1}U_{n-1} - U_n^2,$$

and similarly for the initial condition matrix:

$$\det\begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = U_2U_0 - U_1^2.$$

Substituting these determinant values back into our main determinant equation, we obtain:

$$(-b)^{n-1}(U_2U_0 - U_1^2) = U_{n+1}U_{n-1} - U_n^2.$$

Rearranging the sides completes the proof of Cassini's identity. □

3.4 Fundamental and Companion Sequences (Lucas-Type Sequences)

In the study of second-order linear recurrences, there are two special sequences that serve as the building blocks for all others. We consider the general recurrence relation:

$$X_n = aX_{n-1} + bX_{n-2}, \qquad \text{for } n \geq 2,$$

where a and b are constants (with $b \neq 0$).

Definition 3.9 (Fundamental and Companion Sequences).
The **fundamental sequence** (or Lucas sequence of the first kind), denoted (U_n), satisfies the recurrence with the initial conditions:

$$U_0 = 0, \quad U_1 = 1.$$

The **companion sequence** (or Lucas sequence of the second kind), denoted (V_n), satisfies the exact same recurrence but with the initial conditions:

$$V_0 = 2, \quad V_1 = a.$$

These two sequences are deeply interconnected. The following theorem shows how to generate the companion sequence directly from the fundamental sequence.

Theorem 3.10 (Interrelation Identity).
For any integer $n \geq 1$, the fundamental sequence (U_n) and its companion sequence (V_n) satisfy the identity:

$$V_n = U_{n+1} + bU_{n-1}.$$

Proof. We proceed by strong induction on n.
Base Cases: For $n = 1$, we compute the right-hand side using the initial conditions:

$$U_2 + bU_0 = (aU_1 + bU_0) + b(0) = a(1) + 0 = a.$$

Since $V_1 = a$, the formula holds for $n = 1$.
For $n = 2$, the companion sequence gives $V_2 = aV_1 + bV_0 = a^2 + 2b$. Using the identity formula, we get:

$$U_3 + bU_1 = (aU_2 + bU_1) + b(1) = a(a) + b + b = a^2 + 2b.$$

The formula holds for $n = 2$.
Inductive Step: Assume the relation holds for all integers up to some $n \geq 2$. We want to show it holds for $n + 1$, meaning $V_{n+1} = U_{n+2} + bU_n$.
By the recursive definition of (V_n), we have:

$$V_{n+1} = aV_n + bV_{n-1}.$$

Substituting our inductive hypothesis for V_n and V_{n-1}:

$$V_{n+1} = a(U_{n+1} + bU_{n-1}) + b(U_n + bU_{n-2}).$$

We rearrange the terms to group the parts of the fundamental sequence:

$$V_{n+1} = (aU_{n+1} + bU_n) + b(aU_{n-1} + bU_{n-2}).$$

By the recursive definition of (U_n), we know that $aU_{n+1} + bU_n = U_{n+2}$ and $aU_{n-1} + bU_{n-2} = U_n$. Substituting these yields:

$$V_{n+1} = U_{n+2} + bU_n.$$

This completes the induction. The identity holds for all $n \geq 1$. □

Theorem 3.11 (Binet's Formulas)**.**
Let α and β be the distinct roots of the characteristic equation $t^2 - at - b = 0$. The explicit formulas for the fundamental and companion sequences are given by:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \text{and} \qquad V_n = \alpha^n + \beta^n.$$

Proof. For U_n, applying the general explicit formula derived earlier with $U_0 = 0$ and $U_1 = 1$ directly yields $\frac{\alpha^n - \beta^n}{\alpha - \beta}$.
For V_n, we can use Theorem 3.10 and the fact that $\alpha\beta = -b$ (from Vieta's formulas). Alternatively, applying the general explicit formula with $V_0 = 2$ and $V_1 = a = \alpha + \beta$ gives the result immediately. □

Classical Examples in Number Theory

By choosing specific integer values for a and b, we obtain several famous sequences widely studied in combinatorics and number theory.

Examples.

- **Fibonacci and Lucas Sequences** ($a = 1, b = 1$)**:**

$$F_n = F_{n-1} + F_{n-2}, \quad (F_0 = 0, F_1 = 1) \quad \text{(Fundamental)}$$
$$L_n = L_{n-1} + L_{n-2}, \quad (L_0 = 2, L_1 = 1) \quad \text{(Companion)}$$

- **Pell and Pell-Lucas Sequences** ($a = 2, b = 1$)**:**

$$P_n = 2P_{n-1} + P_{n-2}, \quad (P_0 = 0, P_1 = 1) \quad \text{(Fundamental)}$$
$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad (Q_0 = 2, Q_1 = 2) \quad \text{(Companion)}$$

- **Jacobsthal and Jacobsthal-Lucas Sequences** ($a = 1, b = 2$)**:**

$$J_n = J_{n-1} + 2J_{n-2}, \quad (J_0 = 0, J_1 = 1) \quad \text{(Fundamental)}$$
$$j_n = j_{n-1} + 2j_{n-2}, \quad (j_0 = 2, j_1 = 1) \quad \text{(Companion)}$$

Exercise.
For each of the three classical families defined above, perform the following tasks:

1. **Fundamental Matrix Form:** Show that the fundamental sequence U_n can be generated by the matrix exponentiation:
$$\begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
2. **Companion Matrix Form:** Write down the corresponding initial vector and matrix product that generates the companion sequence vector $\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix}$.
3. **Cassini's Identity:** Apply the general matrix determinant property to deduce the specific Cassini identity for the fundamental sequence U_n.
4. **Binet's Formulas:** Find the characteristic roots α and β, and write down the exact explicit formulas for both the fundamental sequence U_n and its companion V_n.

3.5 Tiling Problem

3.5.1 Linear Case

Consider a tiling problem involving a rectangular board of dimensions $1 \times n$, which consists of n labeled cells:

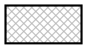
1	2	3	$\cdots$	n-2	n-1	n

$1 \times n$ board with labeled cells

Our objective is to determine the total number of ways to completely tile this board using the following two types of tiles:

- Squares of size 1×1:

- Dominoes of size 1×2:



This problem explores how different combinations of these tiles can fully cover the board, forming the basis for various combinatorial tiling strategies.

Theorem 3.12.
The number of ways to tile a $1 \times n$ board using identical squares and identical dominoes is given by F_{n+1}, where F_n is the Fibonacci sequence.

Proof. Let U_n denote the number of ways to tile a $1 \times n$ board using identical 1×1 squares and identical 1×2 dominoes.

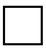
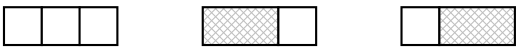
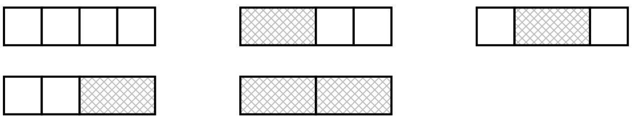
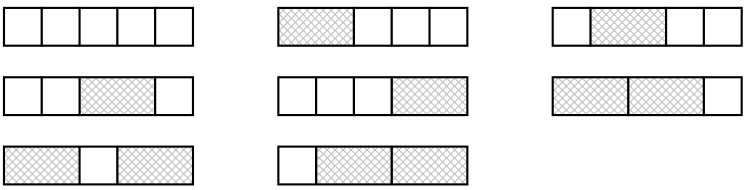
n	**Possible Tilings**
n=1	
n=2	
n=3	
n=4	
n=5	

Table 1: Base Cases for the Linear Tiling Problem

Recursive Case: Consider the last tile placed on a $1 \times n$ board:

- If the last tile is a 1×1 square, the remaining part of the board is a $1 \times (n-1)$ board, which can be tiled in U_{n-1} ways.
- If the last tile is a 1×2 domino, it covers the last two squares, leaving a $1 \times (n-2)$ board, which can be tiled in U_{n-2} ways.

This results in the recurrence relation:

$$U_n = U_{n-1} + U_{n-2}$$

Since this recurrence is identical to that of the Fibonacci sequence, and given the initial conditions

$U_1 = 1$ and $U_2 = 2$, it follows that:

$$U_n = F_{n+1}$$

This completes the proof. □

Theorem 3.13.
The Fibonacci sequence satisfies the identity:

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Proof.

- Induction.
- Consider the problem of counting the number of ways to tile a $1 \times n$ board using indistinguishable 1×2 dominoes and 1×1 squares. We have previously established that F_{n+1}, the $(n+1)$-th Fibonacci number, enumerates all possible tilings of the board.
  To analyze these tilings systematically, we classify them based on the number of dominoes k, which may range from 0 (all squares) to $\left\lfloor \frac{n}{2} \right\rfloor$ (the maximum number of dominoes that can fit in the board). Each domino occupies 2 cells, so using k dominoes covers $2k$ cells, leaving $n - 2k$ cells to be filled with squares.
  The problem of arranging k dominoes and $n - 2k$ squares is equivalent to solving the equation:
  $$x_1 + x_2 + \cdots + x_{k+1} = n - 2k \quad \text{with} \quad x_i \geq 0,$$
  where x_i represents the number of squares between or adjacent to the dominoes. Specifically:
  - x_1: Squares before the first domino
  - x_i $(2 \leq i \leq k)$: Squares between the $(i-1)$-th and i-th domino
  - x_{k+1}: Squares after the last domino

  By the stars and bars theorem, the number of non-negative integer solutions to this equation is:
  $$\binom{(n-2k)+k}{k} = \binom{n-k}{k}.$$
  Summing over all valid values of k gives the total number of tilings:
  $$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$
  The constraint $2k \leq n$ ensures $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, which defines the upper limit of the summation. This combinatorial argument establishes the connection between Fibonacci numbers and domino tilings.

□

3.6 Circular Case

We consider a circular bracelet of length n and investigate the number of ways to tile it using:

- **Identical squares** of size 1×1.
- **Identical dominoes** of size 1×2.

Theorem 3.14.
Let V_n denote the number of ways to tile a circular bracelet of length n using squares and dominoes. Then V_n satisfies the recurrence:

$$V_n = L_n,$$

where L_n is the Lucas number.

Proof. We determine V_n recursively by considering the placement of the last tile.
Base Cases:

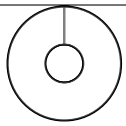
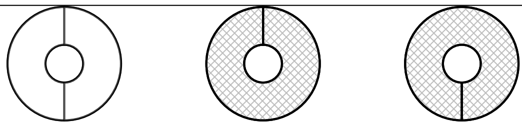
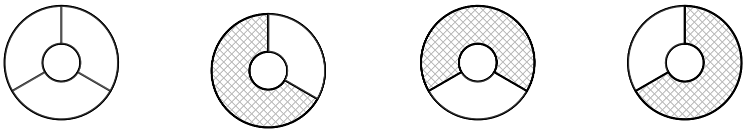
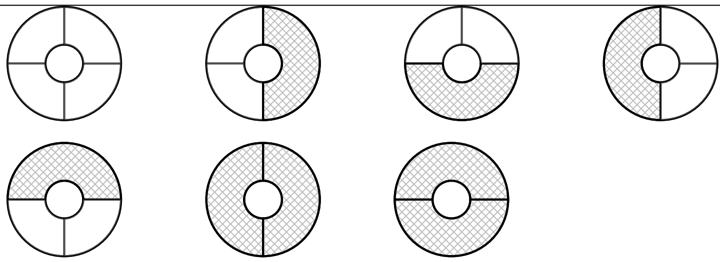
n	**Possible Tilings**
n=1	
n=2	
n=3	
n=4	

Table 2: Base Cases for the Linear Tiling Problem

Recursive Step: Consider a circular bracelet of length n with labeled cells $1, 2, \ldots, n$, where cell n is adjacent to cell 1. Let V_n denote the number of ways to tile this bracelet using indistinguishable 1×1 squares and 1×2 dominoes. We analyze the tiling possibilities through case decomposition:

- **Case 1: Cell 1 contains a square**
  The remaining $n-1$ cells form a linear chain, which can be tiled in F_n ways, where F_n is the n-th Fibonacci number.
- **Case 2: Cell 1 is covered by a domino**
  There are two distinct circular configurations:
  1. Domino covers cells $(1, 2)$
  2. Domino covers cells $(n, 1)$

  Both configurations leave a linear chain of $n-2$ cells, each tiling in F_{n-1} ways. This contributes $2F_{n-1}$ tilings.

Combining both cases yields the recurrence relation:

$$V_n = F_n + 2F_{n-1}.$$

Using the Fibonacci identity $F_{n+1} = F_n + F_{n-1}$, we restructure the equation:

$$V_n = (F_n + F_{n-1}) + F_{n-1} = F_{n+1} + F_{n-1}.$$

This matches the closed-form expression for the Lucas numbers L_n, which satisfy:

$$L_n = F_{n+1} + F_{n-1}.$$

Through induction on n with base cases:

$$V_1 = 1 = L_1$$
$$V_2 = 3 = L_2$$
$$V_3 = 4 = L_3$$

we establish the equivalence for all $n \geq 1$. Therefore, the number of bracelet tilings corresponds to the Lucas sequence:

$$V_n = L_n.$$

□

Theorem 3.15.
The Lucas sequence satisfies the identity:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

Proof. Consider a circular bracelet of n cells. Let $V_n = L_n$ denote the number of tilings using squares and dominoes. We analyze tilings by considering two cases for cell 1:

- *Case 1: Cell 1 contains a square.* The remaining $n-1$ cells form a linear chain, which can be tiled in $\binom{n-1-k}{k}$ ways with k dominoes.
- *Case 2: Cell 1 is covered by a domino.* There are two configurations:
  - Domino covers cells $(1, 2)$
  - Domino covers cells $(n, 1)$

  Each leaves $n-2$ cells in a linear chain, tiling in $2\binom{n-2-(k-1)}{k-1} = 2\binom{n-1-k}{k-1}$ ways.

Combining both cases:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-1-k}{k} + 2\binom{n-1-k}{k-1} \right]$$

Using the binomial identity $\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1}$, we rewrite:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-k}{k} + \binom{n-1-k}{k-1} \right]$$

Notice that:

$$\begin{aligned} \frac{n}{n-k}\binom{n-k}{k} &= \binom{n-k}{k} + \frac{k}{n-k}\binom{n-k}{k} \\ &= \binom{n-k}{k} + \binom{n-1-k}{k-1} \end{aligned}$$

Thus:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-k}{k} + \binom{n-1-k}{k-1} \right] = L_n$$

The final equality follows from matching terms with our initial combinatorial expression. This completes the proof. □

3.7 Tiling and generealized sequence of second order

Linear case

Let consider that we want tile the $1 \times n$ board using squares of 'a' different colors, and dominoes of 'b' different colors,

Theorem 3.16.
The number of ways to tile a $1 \times n$ board using a types of squares and b types of dominoes satisfies both:

1. The linear recurrence:

$$U_n = \begin{cases} 1 & n = 0 \\ a & n = 1 \\ aU_{n-1} + bU_{n-2} & n \geq 2 \end{cases}$$

2. The combinatorial formula:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

Proof.

Recurrence Relation Proof

Consider the last tile in any tiling of a $1 \times n$ board:

- **Case 1: Last tile is a square**
  There are a choices for the square. The remaining $n-1$ cells form a $1 \times (n-1)$ board with U_{n-1} tilings.
  $$\text{Contribution: } aU_{n-1}$$
- **Case 2: Last tile is a domino**
  There are b choices for the domino. The remaining $n-2$ cells form a $1 \times (n-2)$ board with U_{n-2} tilings.
  $$\text{Contribution: } bU_{n-2}$$

Combining both cases gives the recurrence:

$$U_n = aU_{n-1} + bU_{n-2}$$

Combinatorial Proof

Any tiling with k dominoes must contain:

- k dominoes occupying $2k$ cells
- $n-2k$ squares occupying the remaining cells

The number of ways to arrange k dominoes and $n-2k$ squares is equivalent to choosing positions for the dominoes. This is given by the binomial coefficient $\binom{n-k}{k}$, as each domino placement reduces the effective length by k.
Each configuration has:

- a^{n-2k} choices for squares
- b^k choices for dominoes

Summing over all possible k values gives:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

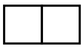
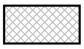
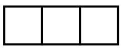
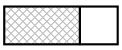
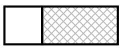
n	Base Case Tilings (a=2, b=3)
1	 (2 ways)
2	 (4 ways)  (3 ways)
3	 (8)  (6)  (6)

Table 3: Example tilings with coefficients $a = 2$, $b = 3$

Both proofs agree through:

$$U_2 = a^2 + b = 2^2 + 3 = 7$$

$$\sum_{k=0}^{1} \binom{2-k}{k} 2^{2-2k} 3^k = \binom{2}{0} 2^2 3^0 + \binom{1}{1} 2^0 3^1 = 4 + 3 = 7$$

This establishes the equivalence between the recursive and combinatorial forms. □

Circular case

Theorem 3.17.
The number of ways to tile a circular bracelet of n labeled cells using a types of squares and b types

of dominoes satisfies:

$$V_n = \begin{cases} a & n = 1 \\ U_n + bU_{n-2} & n \geq 2 \end{cases}$$

where U_n is the linear tiling count from Theorem 1. This can alternatively be expressed as:

$$V_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k + b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Proof.

Case Analysis for Circular Tilings

- **Case 1: Cell 1 contains a square**
  The remaining $n-1$ cells form a linear chain (no circular constraint), yielding:
  $$\text{Contribution: } aU_{n-1}$$
- **Case 2: Cell 1 is covered by a domino**
  Two distinct circular configurations exist:
  - Domino covers cells $(1, 2)$
  - Domino covers cells $(n, 1)$

  Each leaves $n-2$ cells in linear arrangement. Total contribution:
  $$\text{Contribution: } 2bU_{n-2}$$

Combining both cases gives the recurrence:

$$V_n = aU_{n-1} + 2bU_{n-2}$$

Combinatorial Interpretation

From Theorem 1, linear tilings are:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

Circular tilings include:

- *All linear tilings* (U_n)
- *Wrap-around domino tilings* not counted in linear case:
  $$b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Thus:

$$V_n = U_n + bU_{n-2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k + b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Example Verification

For $n = 3$, $a = 2$, $b = 3$:

$$U_1 = 2$$
$$U_2 = 2^2 + 3 = 7$$
$$U_3 = 2 \times 7 + 3 \times 2 = 20$$
$$V_3 = 2 \times 7 + 2 \times 3 \times 2 = 14 + 12 = 26$$
$$\text{Combinatorial: } (20) + 3 \times (2) = 26$$

□

4. Higher-Order Recurrence Relations

Let's consider the homogeneous linear recurrence relation of order m defined by

$$U_n = \begin{cases} a_1U_{n-1} + a_2U_{n-2} + \cdots + a_mU_{n-m}; & n \geq m \\ U_1 = 1; U_0 = U_{-1} = \cdots = U_{-(m-2)} = 0; & -(m-2) \leq n \leq 1 \end{cases}$$

Theorem 4.1.
The $(n+1)$-th term of the homogeneous linear recurrence relation of order m defined above is given by the formula:

$$U_{n+1} = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}.$$

Proof. Exercise: Induction. □