Chapter 3: Introduction to Graph Theory

1. Introduction: The Königsberg Bridge Problem

Graph theory started with a famous problem. The city of Königsberg (now Kaliningrad, Russia) had seven bridges connecting two islands and two mainland areas. The problem asked: is it possible to start at one point, cross each bridge exactly once, and return to the starting point?

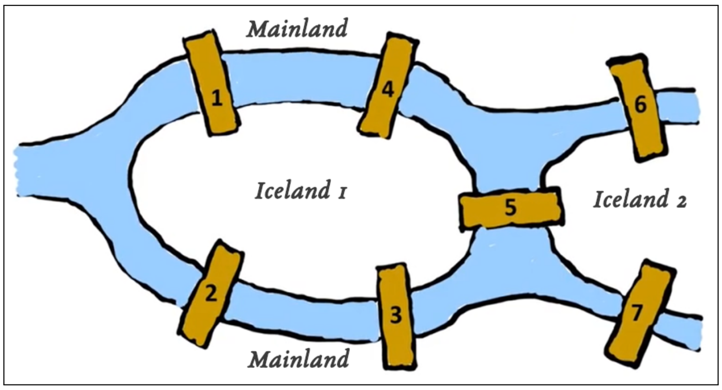


Figure 4: The seven bridges of Königsberg.

We can simplify this problem by drawing each land area as a point (a **vertex**) and each bridge as a line (an **edge**). This drawing is called a **graph**.

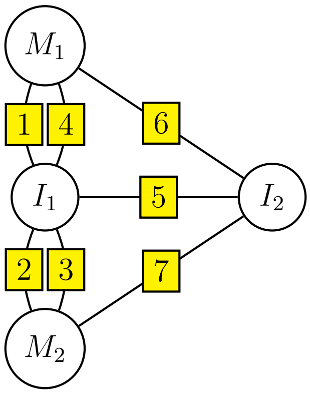


Figure 5: Graph representation of the Königsberg bridges. Vertices represent land areas, edges represent bridges.

Using this graph, we can study the problem mathematically. This chapter introduces the basic ideas of graph theory.

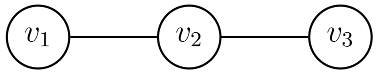
2. What is a Graph?

Definition 2.1 (Graph).
A **graph** $G = (V, E)$ consists of:

- A finite set $V = \{v_1, v_2, \ldots, v_n\}$ called **vertices** (or nodes).
- A finite set $E = \{e_1, e_2, \ldots, e_m\}$ called **edges**. Each edge connects two vertices.

Vertices are drawn as points, edges as lines.

Example.
Let $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_2), (v_2, v_3)\}$. The graph is:

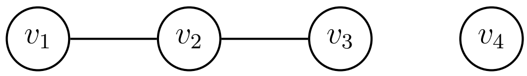


2.1 Basic Graph Terms

Definition 2.2 (Adjacent, Incident, Isolated).
Let $G = (V, E)$ be a graph.

- Two vertices are **adjacent** if an edge connects them.
- Two edges are **adjacent** if they share a common vertex.
- An edge is **incident** to a vertex if the vertex is one of its endpoints.
- A vertex is **isolated** if it has no incident edges.

Example.
Consider the graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_3)\}$:



- v_1 and v_2 are adjacent.
- Edges (v_1, v_2) and (v_2, v_3) are adjacent.
- Edge (v_1, v_2) is incident to vertices v_1 and v_2.
- Vertex v_4 is isolated.

Definition 2.3 (Order and Size).
The **order** of a graph G, denoted $\text{ord}(G)$, is its number of vertices: $\text{ord}(G) = |V|$.
The **size** of a graph G, denoted $e(G)$, is its number of edges: $e(G) = |E|$.

Example.
In the example above, $\text{ord}(G) = 4$ and $e(G) = 2$.

2.2 Degree of a Vertex

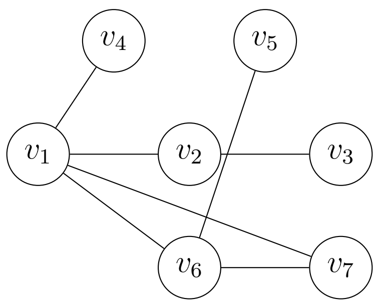
Definition 2.4 (Degree).
The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v.

- The **maximum degree** in G is $\Delta(G) = \max_{v \in V} \deg(v)$.
- The **minimum degree** in G is $\delta(G) = \min_{v \in V} \deg(v)$.

Definition 2.5 (Neighbor and Neighborhood).
Let v be a vertex in a graph $G = (V, E)$.

- A **neighbor** of v is any vertex adjacent to v.
- The **open neighborhood** of v, denoted $N(v)$, is the set of all neighbors of v.
- The **closed neighborhood** of v, denoted $N[v]$, is $N(v) \cup \{v\}$.

Example.
Consider the graph:



Degrees: $\deg(v_1) = 4$, $\deg(v_2) = 1$, $\deg(v_3) = 0$, $\deg(v_7) = 2$. So $\Delta(G) = 4$, $\delta(G) = 0$.

2.3 Special Types of Graphs

Definition 2.6 (Simple Graph).
A **simple graph** has no loops (edges from a vertex to itself) and no multiple edges between the same pair of vertices.

Definition 2.7 (Empty Graph).
An **empty graph** has no edges. It can have any number of vertices.

Definition 2.8 (Complete Graph).
A **complete graph** on n vertices, denoted K_n, is a simple graph where every pair of distinct vertices is connected by an edge.

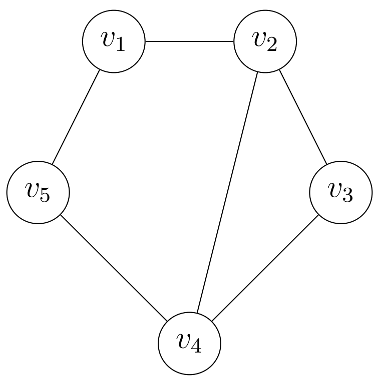
Note: Unless stated otherwise, all graphs in this chapter are assumed to be simple graphs.

3. Directed and Undirected Graphs

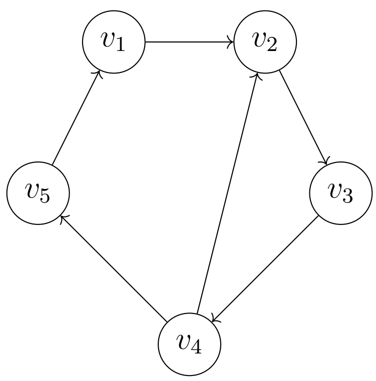
Definition 3.1 (Undirected Graph).
An **undirected graph** has edges with no direction. An edge between u and v is denoted $\{u, v\}$ and can be traversed both ways.

Definition 3.2 (Directed Graph).
A **directed graph** (digraph) has edges with direction. An edge from u to v is denoted (u, v) and can only be traversed from u to v.

Example.
Undirected graph:



Directed graph:



Note: In this chapter, we focus only on undirected graphs.

4. The Handshaking Lemma

Lemma 4.1 (Handshaking Lemma).
For any simple graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Theorem 4.2.
In any simple graph, the number of vertices with odd degree is even.

5. Walks, Trails, Paths, Circuits, and Cycles

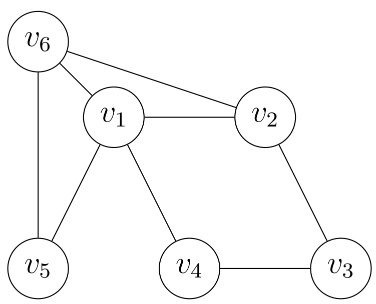
Definition 5.1.
Let $G = (V, E)$ be a simple graph.

- A **walk** is a sequence of vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_k}$ where each consecutive pair is adjacent i. e. $\left(v_{i_j}, v_{i_{j+1}}\right) \in E$. The **length** of a walk is the number of its edges (k).
- A **trail** is a walk with no repeated edges.
- A **path** is a walk with no repeated vertices (and hence no repeated edges).
- A **circuit** is a closed trail (starts and ends at same vertex).
- A **cycle** is a closed path (only the start/end vertex repeats).

Term	Repeated Vertices?	Repeated Edges?	Closed?
Walk	Yes	Yes	No
Trail	Yes	No	No
Path	No	No	No
Circuit	Yes	No	Yes
Cycle	No (except start/end)	No	Yes

Table 2: Comparison of walk, trail, path, circuit, and cycle.

Example.
Consider the graph G:



- **Walk:** v_1, v_2, v_1, v_4
  (Repeats edge (v_1, v_2) → Not a trail).
- **Trail:** v_5, v_1, v_6, v_2, v_1
  (Repeats vertex v_1, but no edges repeated → Not a path).
- **Path:** v_5, v_6, v_2
  (No repeated vertices).
- **Circuit:** $v_1, v_5, v_6, v_1, v_2, v_6, v_1$
  (Closed trail. Visits v_1 in the middle → Not a cycle).

- **Cycle:** v_1, v_6, v_5, v_1
  (closed path).

Lemma 5.2.
In any simple graph with at least two vertices, there are at least two vertices with the same degree.

Theorem 5.3.
Let $G = (V, E)$ be a simple graph, and let $u, v \in V$ be two distinct vertices. Any u-v walk in G contains a u-v path in G.

6. Complement of a Graph

Definition 6.1 (Complement).
The **complement** of a simple graph G, denoted $\overline{G}$, has the same vertices. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Graph G Complement $\overline{G}$

Example.

Exercise.
For a simple graph G with n vertices, show that $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.

Solution.
In a simple graph with n vertices, the maximum number of edges is $\binom{n}{2}$. The complement $\overline{G}$ contains exactly the edges not in G. Therefore,

$$|E(G)| + |E(\overline{G})| = \binom{n}{2}.$$

7. Subgraphs

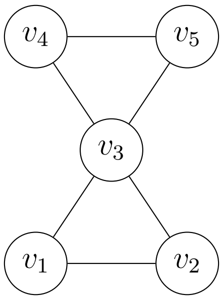
Definition 7.1 (Subgraph).
A graph $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$, and every edge in E' has both endpoints in V'.

Definition 7.2 (Spanning Subgraph).
A **spanning subgraph** of G is a subgraph with the same vertex set as G.

Definition 7.3 (Induced Subgraph).
An **induced subgraph** by a vertex set $V' \subseteq V$ contains all edges of G with both endpoints in V'.

Definition 7.4 (Edge-Induced Subgraph).
An **edge-induced subgraph** by an edge set $E' \subseteq E$ contains exactly the edges in E' and all vertices incident to them.

Example.
Consider the graph G:

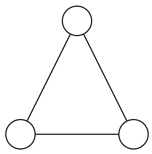


- Subgraph: e.g., vertices $\{v_1, v_2, v_3\}$ with edges $\{(v_1, v_2), (v_2, v_3)\}$.
- Spanning subgraph: keep all vertices, remove some edges.
- Induced subgraph by $\{v_1, v_2, v_3\}$: includes all edges between these vertices.
- Edge-induced subgraph by $\{(v_1, v_2), (v_3, v_4)\}$: includes vertices v_1, v_2, v_3, v_4 and those edges.

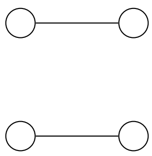
8. Connectivity and Components

Definition 8.1 (Connected Graph).
A simple graph is **connected** if there is a path between every pair of vertices. Otherwise, it is **disconnected**.

Example.
Connected:

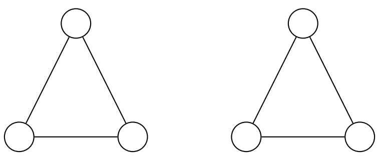


Disconnected:



Definition 8.2 (Component).
A **component** of a simple graph is a maximal connected subgraph. A graph is connected if it has exactly one component.

Example.
The graph below has two components (two triangles).

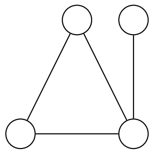


9. Cliques and Stable Sets

Definition 9.1 (Stable Set).
A **stable set** (independent set) in a simple graph is a set of vertices with no edges between them. The size of a largest stable set is the **stability number** $\alpha(G)$.

Definition 9.2 (Clique).
A **clique** in a simple graph is a set of vertices where every pair is adjacent. The size of a largest clique is the **clique number** $\omega(G)$.

Example.
In the graph:



Maximal cliques: $\{v_1, v_2, v_3\}$ and $\{v_2, v_4\}$. Maximum clique size: 3.

10. Trees and Forests

Definition 10.1 (Tree).
A **tree** is a connected simple graph with no cycles.

Definition 10.2 (Forest).
A **forest** is a simple graph with no cycles (each component is a tree).

Definition 10.3 (Leaf).
A **leaf** is a vertex of degree 1 in a tree.

Definition 10.4 (Spanning Tree).
A **spanning tree** of a connected simple graph G is a spanning subgraph that is a tree.

Example.
Examples of trees:

- Single vertex.
- Path graph.
- Star graph.
- Binary tree.

Theorem 10.5.
A simple graph is a tree if and only if there is exactly one path between every pair of vertices.

Theorem 10.6.
A tree with n vertices has $n - 1$ edges.

Theorem 10.7.
Let T be a tree with n_i vertices of degree i, where $i = 1, 2, \ldots, k$ and $k = \Delta(T)$. Then:

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k.$$

Theorem 10.8.
A forest with n vertices and k trees has $n - k$ edges.