Chapter 2: Advanced Counting Principles

In the first chapter, we covered the foundational concepts and rules of counting. Now, we will explore more advanced counting principles that allow us to tackle more complex problems.

1. Cardinality of Sets

Let S be a finite set, and let A and B be subsets of S.

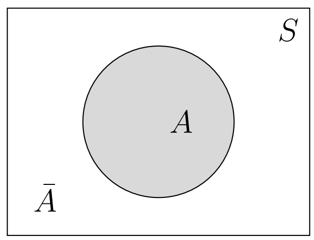
Definition 1.1.
The *cardinality* of a set S, denoted $|S|$, is the number of elements in S. Specifically, if $S = \{e_1, e_2, \ldots, e_n\}$, then $|S| = n$.

Definition 1.2 (Complement of A)**.**
The complement of a subset A of S, denoted $\bar{A}$, consists of all elements of S that are not in A:

$$\bar{A} = \{x \in S \mid x \notin A\}.$$

Cardinality: The cardinality of the complement $\bar{A}$ is given by

$$|\bar{A}| = |S| - |A|.$$

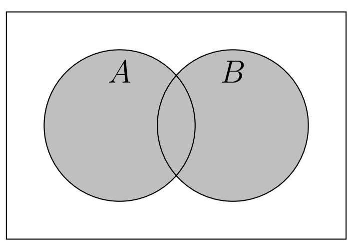


Definition 1.3 (Union of Two Sets)**.**
The *union* of two subsets A and B of S, denoted by $A \cup B$, is the set of all elements that are in A, in B, or in both:

$$A \cup B = \{x \in S \mid x \in A \text{ or } x \in B\}.$$

Cardinality: The cardinality of $A \cup B$ is calculated by

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

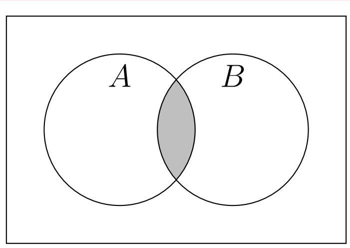


Definition 1.4 (Intersection of Two Sets)**.**
The *intersection* of two subsets A and B of S, denoted by $A \cap B$, is the set of all elements that are in both A and B:

$$A \cap B = \{x \in S \mid x \in A \text{ and } x \in B\}.$$

Cardinality: The cardinality of $A \cap B$ is simply

$$|A \cap B|.$$

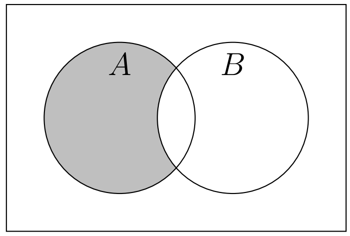


Definition 1.5 (Difference of Two Sets).
The *difference* of two sets A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B:

$$A \setminus B = \{x \in S \mid x \in A \text{ and } x \notin B\}.$$

Cardinality: The cardinality of $A \setminus B$ is

$$|A \setminus B| = |A| - |A \cap B|.$$



1.1 Finite Unions and Intersections

Now, let $A_1, A_2, \ldots, A_n$ be subsets of S.

Definition 1.6 (Finite Unions).
The *finite union* of $A_1, A_2, \ldots, A_n$, subsets of S, is defined as:

$$\bigcup_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for at least one } i\}.$$

Cardinality: The cardinality of $\bigcup_{i=1}^{n} A_i$ is determined by the inclusion-exclusion principle, which leads to a more complex formula.

Definition 1.7 (Finite Unions of Disjoint Subsets).
If $A_1, A_2, \ldots, A_n$ are subsets of S such that each pairwise intersection is empty, i.e.,

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j,$$

then their union is denoted by $\bigoplus_{i=1}^{n} A_i$.
Cardinality: The cardinality of $\bigoplus_{i=1}^{n} A_i$ is simply the sum of the cardinalities of the individual sets:

$$\left|\bigoplus_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

Definition 1.8 (Finite Intersections).
The *finite intersection* of $A_1, A_2, \ldots, A_n$, subsets of S, is defined as:

$$\bigcap_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for all } i\}.$$

Cardinality: The cardinality of $\bigcap_{i=1}^{n} A_i$ is simply

$$\left|\bigcap_{i=1}^{n} A_i\right|.$$

Definition 1.9 (Power Set).
The *power set* of a set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Proposition 1.10 (Cardinality of the Power Set).
If $|S| = n$, the cardinality of the power set $\mathcal{P}(S)$ is given by:

$$|\mathcal{P}(S)| = 2^n.$$

Proof. To count the total number of subsets of a set S with n elements, we observe that subsets can have cardinalities ranging from 0 to n. The number of subsets with exactly k elements is given by $\binom{n}{k}$, the number of ways to choose k elements from n.
By the *addition principle*, the total number of subsets is:

$$|\mathcal{P}(S)| = \sum_{k=0}^{n} \binom{n}{k}.$$

Using the binomial theorem:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Thus, the total number of subsets of S is:

$$|\mathcal{P}(S)| = 2^n.$$

□

1.2 Floor and Ceiling Functions

Definition 1.11 (Floor and Ceiling Functions).
For any real number x, we know that x lies between two integers n and $n+1$, where $n \leq x < n+1$. In this case:

- The integer n is denoted by $\lfloor x \rfloor$, called the *floor function*, which represents the greatest integer less than or equal to x.
- The integer $n+1$ is denoted by $\lceil x \rceil$, called the *ceiling function*, which represents the smallest integer greater than or equal to x.

2. Pigeonhole Principle (Dirichlet box principle)

Theorem 2.1 (Pigeonhole Principle - Version 1).
If $n+1$ or more pigeons are placed into n pigeonholes, then at least one pigeonhole must contain more than one pigeon.

Proof. Assume, for the sake of contradiction, that no pigeonhole contains more than one pigeon. Then, each of the n pigeonholes contains at most one pigeon, leading to a maximum of n pigeons. However, we have $n+1$ pigeons, which is a contradiction. Therefore, at least one pigeonhole must contain more than one pigeon. □

Example.

- In a group of 8 students, at least two of them must have the same day of the week as their birthday.
  To solve this using the Pigeonhole Principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (8 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the Pigeonhole Principle (PHP), with 8 pigeons and only 7 pigeonholes, at least one day must be shared by two students. Thus, at least two students have the same birthday day of the week.
- In a group of 13 students, at least two of them must have the same birth month.
  To solve this using the Pigeonhole Principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (13 students).
  - **Pigeonholes (boxes):** The months of the year (12 months).

  By the Pigeonhole Principle (PHP), with 13 pigeons and only 12 pigeonholes, at least one month must be shared by two students. Thus, at least two students have the same birth month.

Theorem 2.2 (Pigeonhole Principle - Version 2).
If $kn + 1$ pigeons are placed into n pigeonholes, where k is a positive integer, then at least one pigeonhole must contain at least $k + 1$ pigeons.

Proof. Assume, for the sake of contradiction, that each pigeonhole contains at most k pigeons. Then the total number of pigeons would be kn. Since there are $kn + 1$ pigeons, this contradicts our assumption. Hence, at least one pigeonhole must contain at least $k + 1$ pigeons. □

Example.

- In a group of 22 students, at least 3 of them must have the same day of the week as their birthday.
  To solve this using the Pigeonhole Principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (22 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the Pigeonhole Principle (PHP), with 22 pigeons and only 7 pigeonholes, at least one day must be shared by at least two students. Thus, at least two students have the same birthday day of the week.
- In a group of 37 people, at least 4 must have been born in the same month.
  To solve this using the Pigeonhole Principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each person in the group (37 people in total).
  - **Pigeonholes (boxes):** The months of the year (12 months).

  By the Pigeonhole Principle (PHP), with 37 pigeons and only 12 pigeonholes, at least one month must be shared by at least 4 people. Thus, at least 4 people must have been born in the same month.

Theorem 2.3 (Generalized Pigeonhole Principle - Version 3).
If m pigeons are placed into n pigeonholes, then at least one pigeonhole contains at least $\lceil \frac{m}{n} \rceil$ pigeons.

Proof. Assume, for the sake of contradiction, that every pigeonhole contains fewer than $\lceil \frac{m}{n} \rceil$ pigeons. Then the total number of pigeons would be less than $n \times \lceil \frac{m}{n} \rceil$, which contradicts the fact that there are m pigeons. Therefore, at least one pigeonhole must contain at least $\lceil \frac{m}{n} \rceil$ pigeons. □

Example.

Suppose there are 316 students in the first year of NHSM, and we want to distribute them into 12 groups, at least one group must contain at least 27 students.
To solve this using the generalized Pigeonhole Principle, we define the "pigeons" and "pigeonholes" as follows:

- **Pigeons:** The 316 students in total.
- **Pigeonholes (boxes):** The 12 groups.

By the generalized Pigeonhole Principle (PHP), with 316 students and only 12 groups, at least one group must contain at least $\lceil \frac{316}{12} \rceil = 27$ students. Thus, at least one group must have at least 27 students.

Exercise.
Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should we grab to ensure we get a pair of the same color?

Solution.
To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the Pigeonhole Principle, grabbing 5 gloves ensures at least two gloves of the same color, since there are only 4 colors.
Thus, we need to grab at least 5 gloves.

Exercise.
Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should an octopus (with 8 hands) grab to ensure it gets a pair of the same color?

Solution.
To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the Pigeonhole Principle, even if the octopus grabs $7 \times 4 + 1$ gloves, it ensures that at least 8 gloves of the same color are taken, since there are only 4 colors.
Thus, the octopus needs to grab at least 29 gloves.

Remarks.
- The Pigeonhole Principle can be used to prove the existence of certain properties within a set of objects.
- Applying the Pigeonhole Principle is not always straightforward and may require thoughtful construction of "pigeons" and "holes."
- A well-constructed approach can lead to a concise and elegant proof.
- The Pigeonhole Principle guarantees that there is a *certain* box that contains at least two objects. However, it does not tell us *which* box it is or *which* objects it contains.

Exercise.
Show that in a set $S = \{a_1, a_2, \ldots, a_{n+1}\}$ of $n+1$ integers, there are at least two integers whose difference is divisible by n.

3. Inclusion-Exclusion Principle

Let S be a finite set, and $A_1, A_2 \subseteq S$. To compute $|A_1 \cup A_2|$, we sum the cardinalities of the individual sets:

$$|A_1| + |A_2|.$$

However, elements in $A_1 \cap A_2$ are counted twice, so we subtract the intersection:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2}$ (the elements not in $A_1 \cup A_2$) is:

$$|\overline{A_1} \cap \overline{A_2}| = |S \setminus (A_1 \cup A_2)| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

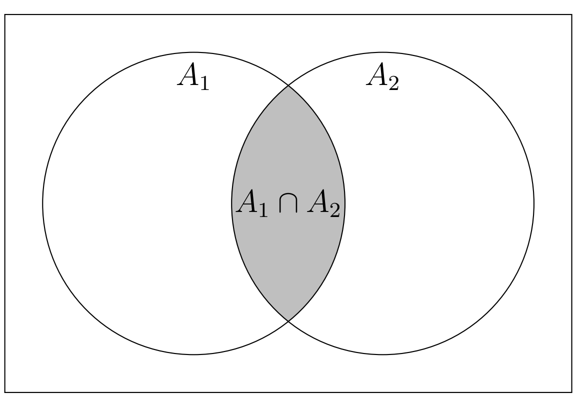


Figure 2: Venn diagram of two sets

3.1 Case of the three subsets

Let S now be a finite set, and $A_1, A_2, A_3 \subseteq S$. To compute $|A_1 \cup A_2 \cup A_3|$, we begin by summing the cardinalities of the individual sets:

$$|A_1| + |A_2| + |A_3|.$$

However, elements in the pairwise intersections $A_1 \cap A_2$, $A_1 \cap A_3$, and $A_2 \cap A_3$ are counted twice, so we subtract the sizes of these intersections:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|.$$

But now, the elements in the triple intersection $A_1 \cap A_2 \cap A_3$ have been subtracted three times, so we add back the size of this triple intersection:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ (the elements not in $A_1 \cup A_2 \cup A_3$) is given by:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S \setminus (A_1 \cup A_2 \cup A_3)| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

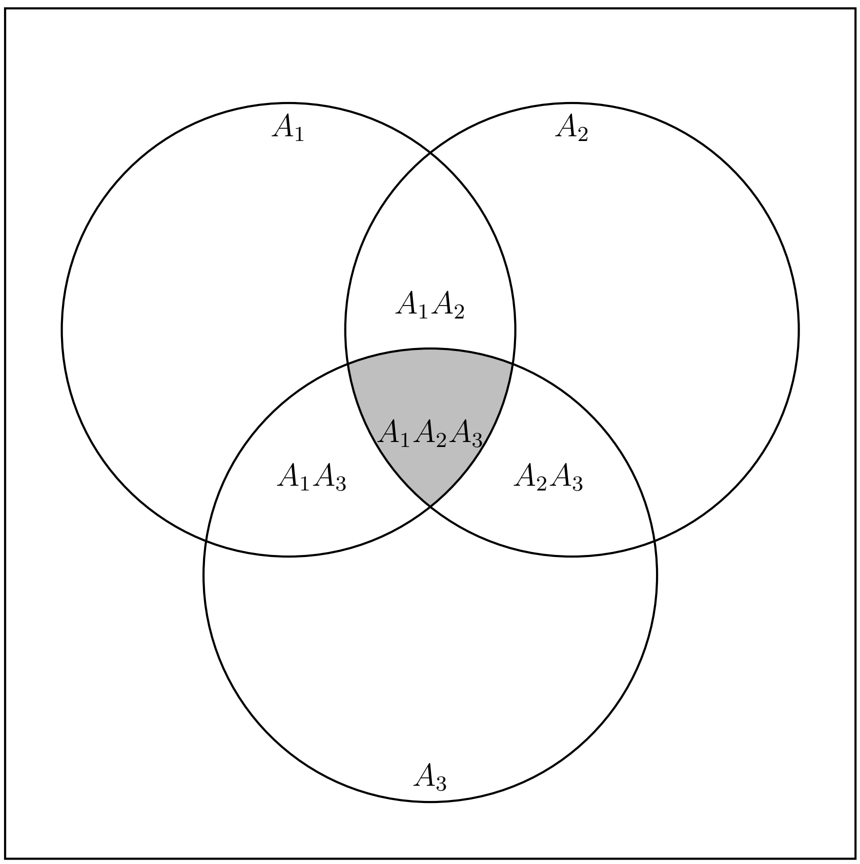


Figure 3: Venn diagram of three sets

3.2 General case

As we observe in the cases of two sets and three sets, the computation of the union always starts with the **inclusion** of the cardinalities of the individual sets. Next, we **exclude** the intersections of pairs of sets because these elements are counted multiple times. This process of alternating **inclusion-exclusion** continues, giving rise to the concept known as the **Inclusion-Exclusion Principle (IEP)**.

In the following theorem, we will generalize this principle to compute the cardinality of the union of n subsets of a finite (universal) set S.

Before proceeding, let us consider subsets $A_1, A_2, \ldots, A_n \subseteq S$, where each A_i represents the set of elements satisfying property i (for $i = 1, 2, \ldots, n$). To simplify notation, we denote the intersection $A_i \cap A_j$ directly as A_iA_j. For example, $A_1A_2A_3$ represents $A_1 \cap A_2 \cap A_3$.

Theorem 3.1 (General Inclusion-Exclusion Principle)**.**
Let $A_1, A_2, \ldots, A_n$ be subsets of a finite universal set S. The cardinality of the union $|A_1 \cup A_2 \cup \cdots \cup A_n|$ is given by the following formula:

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\
&\quad - \cdots + (-1)^{n-1} \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}| \\
&= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.
\end{aligned}$$

Proof. First proof: By induction.

Second proof:
The theorem calculates the number of elements in the finite union of subsets $A_1, A_2, \ldots, A_n$. The key idea is that for any element $x \in S$, it is either counted 0 times (if $x \notin A_1 \cup A_2 \cup \cdots \cup A_n$) or counted exactly once (if $x \in A_1 \cup A_2 \cup \cdots \cup A_n$).

- If $x \notin A_1 \cup A_2 \cup \cdots \cup A_n$: This implies that $x \notin A_i$ for any $1 \le i \le n$. Consequently, x is not in any intersection of the subsets, and thus the right-hand side (RHS) equals the left-hand side (LHS).
- If $x \in A_1 \cup A_2 \cup \cdots \cup A_n$: This means that x satisfies at least one of the properties $1, 2, \ldots, n$. Suppose x satisfies exactly k properties, corresponding to the indices $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$. Then:
  - x is included $\binom{k}{1} = k$ times in any single subset of $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$.
  - It is excluded $\binom{k}{2}$ times in any intersection of two subsets of $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$.
  - It is included $\binom{k}{3}$ times in any intersection of three subsets of $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$.
  - $\cdots$
  - Finally, it is included (if k is odd) or excluded (if k is even) in the intersection of k subsets of $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$.

  Therefore, the right-hand side counts x:
  $$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k} = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \quad \text{times.}$$
  But:
  $$\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} = -\sum_{i=1}^{k} (-1)^{i} \binom{k}{i} = -\left(\sum_{i=0}^{k} (-1)^{i} \binom{k}{i} - 1\right) = 1.$$
  So, any x is counted exactly once, meaning the formula correctly calculates the union of the subsets $A_1, A_2, \ldots, A_n$.

□

Corollary 3.2 (Complementary form)**.**
The cardinality of the complement of the union of subsets $A_1, A_2, \ldots, A_n$ is:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

Proof. By definition, the complement of the union is:
$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$
Substituting the Inclusion-Exclusion formula for $|A_1 \cup A_2 \cup \cdots \cup A_n|$ proves the result. □

When to Use IEP

- **Union of Sets:** Apply IEP to calculate the size of a union when overlaps cause over-counting.
- **Complement of Union:** To compute the complement of a union, subtract the union's size from the size of the universal set, and this is the common application of IEP.
- The universal set S includes all the elements under consideration for the problem, without considering any specific conditions or restrictions imposed by subsets.

Steps for Applying IEP

1. **Define the Sets:** Specify the subsets $A_1, A_2, \ldots, A_n$, where each represents a specific condition or property.
2. **Compute Intersections:** Calculate the sizes of all intersections, such as:
$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| .$$
3. **Apply the IEP Formula:** Compute the union's size using:
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}| .$$

4. Classical examples of using IEP

We now apply the principle to a series of classical combinatorial problems.

4.1 Derangements: Permutations Without Fixed Points

Definition 4.1 (Derangement).
A **derangement** is a permutation of n elements $\{e_1, e_2, \ldots, e_n\}$ such that no element e_i is in its original position i.

Theorem 4.2 (Number of derangements).
The number of derangements of a set of n elements, denoted by D_n, is given by:
$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

From the definition, a derangement of $\{e_1, \ldots, e_n\}$ is a permutation where e_1 is not in position 1, e_2 is not in position 2, and so on, with e_n not in position n. To calculate the number of such permutations, we count the number of elements in the set of permutations that satisfy $e_1 \notin$ position $1, e_2 \notin$ position $2, \ldots, e_n \notin$ position n. This requires counting the intersections of sets, which leads us to the complementary form of Inclusion-Exclusion Principle (IEP):

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}| .$$

- **Define the Sets:**
  - $\overline{A_i}$: The set of permutations where e_i is not in position i.
  - A_i: The set of permutations where e_i is in position i.
  - S: The universal set, representing all permutations of $\{e_1, \ldots, e_n\}$, without any restrictions.
- **Compute Intersections:** Using the definition of intersections:
  - Cardinality of S:
$$|S| = n!.$$
  - Single Sets: Each $|A_{i_1}| = (n-1)!$, so:
$$\sum_{1 \leq i_1 \leq n} |A_{i_1}| = \binom{n}{1} (n-1)!.$$

– Intersections of Two Sets: The intersection $|A_{i_1} \cap A_{i_2}| = (n-2)!$, so:

$$\sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| = \binom{n}{2}(n-2)!.$$

– General Case for k Sets: The intersection of k subsets $|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$. Thus:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = \binom{n}{k}(n-k)!.$$

- **Apply the IEP Formula:** Substituting these results into the Inclusion-Exclusion formula:

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= |S| - \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|, \\
&= n! - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)!, \\
&= n! + \sum_{k=1}^{n}(-1)^{k}\frac{n!}{k!}, \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$

This proves the formula for the number of derangements $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

4.2 Number of Integer Solutions to a Linear Equation with Constraints

We aim to count the number of solutions to the equation (S), defined as:

$$(S) \quad \begin{cases} x_1 + x_2 + \cdots + x_n = k \\ r_i \le x_i \le s_i \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$

First, let us define new variables $y_i = x_i - r_i$ for each $i \in \{1, 2, \cdots, n\}$. Substituting these into the equation, we obtain the equivalent system:

$$(S) \quad \begin{cases} y_1 + y_2 + \cdots + y_n = k - \underbrace{\sum_{i=1}^{n} r_i}_{r} \\ 0 \le y_i \le \underbrace{s_i - r_i}_{l_i} \quad \forall i \in \{1, 2, \cdots, n\} \end{cases} \iff \begin{cases} y_1 + y_2 + \cdots + y_n = k - r \\ 0 \le y_i \le l_i \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$

Next, consider the equation (Ω) obtained by relaxing the upper bounds on y_i, i.e., without the constraints on y_i:

$$(\Omega) \quad \begin{cases} y_1 + y_2 + \cdots + y_n = k - r \\ y_i \ge 0 \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$

Our task is to find the number of solutions to (Ω) under the constraints $0 \le y_1 \le l_1$, $0 \le y_2 \le l_2$, and so on for each y_i. This is equivalent to counting the size of the intersection of sets:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$$

where

$$\overline{A_i} = \{(y_1, y_2, \cdots, y_n) \in \Omega : 0 \le y_i \le l_i\} \quad \forall i \in \{1, 2, \ldots, n\}.$$

Using the Inclusion-Exclusion Principle (IEP), we have:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |\Omega| - \sum_{m=1}^{n}(-1)^{m-1} \sum_{1 \le i_1 < \cdots < i_m \le n} |A_{i_1} \cap \cdots \cap A_{i_m}|,$$

where

$$A_i = \{(y_1, y_2, \cdots, y_n) \in \Omega : y_i \ge l_i + 1\} \quad \forall i \in \{1, 2, \ldots, n\}.$$

Now, to find the number of solutions for each A_i, we make a substitution: let $z_i = y_i - l_i - 1$ for each i, so that $y_i \ge l_i + 1$ corresponds to $z_i \ge 0$. The number of solutions to A_i is then:

$$|A_i| = \left(\!\!\binom{n}{k-r-l_i-1}\!\!\right).$$

For the intersection of two sets A_i and A_j, the number of solutions is:

$$|A_i \cap A_j| = \left(\!\!\binom{n}{k-r-l_i-l_j-2}\!\!\right).$$

In the general case, the number of solutions for the intersection of m sets $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ is:

$$|A_{i_1} \cap \cdots \cap A_{i_m}| = \left(\!\!\binom{n}{k-r-l_{i_1}-l_{i_2}-\cdots-l_{i_m}-m}\!\!\right).$$

Finally, the cardinality of Ω is:

$$|\Omega| = \left(\!\!\binom{n}{k-r}\!\!\right).$$

Substituting these values into the Inclusion-Exclusion formula, we obtain the number of solutions to the original equation under the given constraints:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \left(\!\!\binom{n}{k-r}\!\!\right) - \sum_{m=1}^{n} (-1)^{m-1} \sum_{1 \le i_1 < \cdots < i_m \le n} \left(\!\!\binom{n}{k-r-l_{i_1}-l_{i_2}-\cdots-l_{i_m}-m}\!\!\right).$$

4.3 IEP and Euler's Totient Function

Definition 4.3 (Euler's Totient Function)**.**
The Euler's Totient function, denoted as $\varphi(n)$, counts the number of positive integers x such that $1 \le x \le n$ and $\gcd(x, n) = 1$. That is,

$$\varphi(n) = |\{x \mid 1 \le x \le n,\ \gcd(x, n) = 1\}|$$

where $\gcd(x, n)$ denotes the greatest common divisor of x and n, and the set $\{x \mid 1 \le x \le n,\ \gcd(x, n) = 1\}$ contains all integers from 1 to n that are coprime with n.

Counting directly the integers that are coprime with n is not an efficient method for computing $\varphi(n)$. Instead, we use the **inclusion-exclusion principle** to derive a more efficient formula for $\varphi(n)$.

Any positive integer n has a unique prime factorization of the form:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$$

where $p_1, p_2, \ldots, p_m$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are their respective multiplicities.

To use the inclusion-exclusion principle, we will use the fact that any number that is not coprime with n has at least one factor other than 1 in common with n. So, consider the set of integers less than or equal to n that have the factor p_i in common with n, denoted by A_i. Therefore, the number of integers that are coprime with n is given by:

$$\varphi(n) = n - |A_1 \cup A_2 \cup \cdots \cup A_m|$$

Now, using the inclusion-exclusion principle, we can express the size of the union of these sets as:

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

where $|A_i|$ is the number of integers divisible by p_i, and the other terms account for intersections of the sets.

Counting the Intersections:

- **Single Sets $|A_{i_1}|$:** The set A_{i_1} consists of all integers divisible by p_{i_1}. The number of such integers is:

  $$|A_{i_1}| = \left\lfloor \frac{n}{p_{i_1}} \right\rfloor = \frac{n}{p_{i_1}}$$

  where $\lfloor\rfloor$ is the floor function, which counts the number of multiples of p_{i_1} less than or equal to n.
- **Intersections of Two Sets $|A_{i_1} \cap A_{i_2}|$:** The intersection $A_{i_1} \cap A_{i_2}$ consists of integers divisible

by both p_{i_1} and p_{i_2}. Since p_{i_1} and p_{i_2} are distinct primes, we know that:

$$\operatorname{lcm}(p_{i_1}, p_{i_2}) = p_{i_1}p_{i_2}$$

Therefore, the number of elements in this intersection is:

$$|A_{i_1} \cap A_{i_2}| = \left\lfloor \frac{n}{p_{i_1}p_{i_2}} \right\rfloor = \frac{n}{p_{i_1}p_{i_2}}$$

- **Intersections of Three Sets $|A_{i_1} \cap A_{i_2} \cap A_{i_3}|$:** The intersection $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ consists of integers divisible by the least common multiple of $p_{i_1}, p_{i_2}, p_{i_3}$. Since these are distinct primes, we have:

$$\operatorname{lcm}(p_{i_1}, p_{i_2}, p_{i_3}) = p_{i_1}p_{i_2}p_{i_3}$$

  The number of integers in this intersection is:

$$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \left\lfloor \frac{n}{p_{i_1}p_{i_2}p_{i_3}} \right\rfloor = \frac{n}{p_{i_1}p_{i_2}p_{i_3}}$$

- **General Case $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$:** For any k-tuple $(i_1, i_2, \ldots, i_k)$, the intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ consists of integers divisible by the least common multiple of the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$. Since all $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ are distinct primes, we have:

$$\operatorname{lcm}(p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = p_{i_1}p_{i_2}\ldots p_{i_k}$$

  The number of such integers is:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \left\lfloor \frac{n}{p_{i_1}p_{i_2}\ldots p_{i_k}} \right\rfloor = \frac{n}{p_{i_1}p_{i_2}\ldots p_{i_k}}$$

Using the inclusion-exclusion principle, the final formula for $\varphi(n)$ becomes:

$$\begin{aligned}
\varphi(n) &= n - \sum_{i_1=1}^{m} \frac{n}{p_{i_1}} + \sum_{1 \le i_1 < i_2 \le m} \frac{n}{p_{i_1}p_{i_2}} - \sum_{1 \le i_1 < i_2 < i_3 \le m} \frac{n}{p_{i_1}p_{i_2}p_{i_3}} + \cdots + (-1)^{m-1} \frac{n}{p_1p_2\ldots p_m} \\
&= n \left(1 - \sum_{i_1=1}^{m} \frac{1}{p_{i_1}} + \sum_{1 \le i_1 < i_2 \le m} \frac{1}{p_{i_1}p_{i_2}} - \sum_{1 \le i_1 < i_2 < i_3 \le m} \frac{1}{p_{i_1}p_{i_2}p_{i_3}} + \cdots + (-1)^{m-1} \frac{1}{p_1p_2\ldots p_m} \right) \\
&= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_m}\right) = n \prod_{k=1}^{m} \left(1 - \frac{1}{p_k}\right).
\end{aligned}$$

Example (Numerical example)**.**
We begin by finding the prime factorization of 2024:

$$2024 = 2^3 \times 11 \times 23$$

Now, we apply the formula for Euler's Totient Function:

$$\varphi(2024) = 2024 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{11}\right) \times \left(1 - \frac{1}{23}\right)$$

We simplify each term:

$$1 - \frac{1}{2} = \frac{1}{2}, \quad 1 - \frac{1}{11} = \frac{10}{11}, \quad 1 - \frac{1}{23} = \frac{22}{23}$$

Now, we compute:

$$\varphi(2024) = 2024 \times \frac{1}{2} \times \frac{10}{11} \times \frac{22}{23}$$

First, simplify step by step:

$$2024 \times \frac{1}{2} = 1012$$
$$1012 \times \frac{10}{11} = 920$$
$$920 \times \frac{22}{23} = 880$$

Thus, the value of Euler's Totient Function for 2024 is:

$$\varphi(2024) = 880$$

4.4 The Ménage Problem

The Ménage problem is a classical combinatorics problem that asks:

Given n married couples, how many ways can one arrange these n couples around a circular table such that men and women alternate in seating, and no woman sits next to her husband?

This problem was formulated in 1891 by Édouard Lucas. The first explicit formula for the problem was published by Touchard in 1934, although it lacked a proof. In 1943, Kaplansky provided a proof of Touchard's formula. The formula for the number of valid arrangements, M_n, is:

$$M_n = 2(n!) \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.$$

From the problem statement, we seek the number of alternating arrangements such that no couple is seated together. This translates to finding the cardinality of the set $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$, where $\overline{A_i}$ is the set of configurations in which couple i is not seated together. Using the Inclusion-Exclusion Principle (IEP), we have:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| ,$$

where A_i is the set of configurations in which couple i is seated together, and S is the set of all alternating seating arrangements.

It is straightforward to show that $|S| = 2(n!)^2$. This follows from the fact that women can occupy either odd or even positions, resulting in 2 configurations. For each configuration, the women and men can be independently arranged in $n!$ ways, yielding $2(n!)^2$ total alternating arrangements.

Next, we determine the formula for:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| .$$

This requires analyzing the placement of k dominoes on a circular arrangement.

4.4.1 Placing k Identical, Non-Overlapping Dominoes on m Labeled Circular Positions

To analyze the circular case effectively, we first examine the simpler linear case. Consider k identical dominoes:

,

which we aim to place on a linear grid of m labeled positions such that no two dominoes overlap. Each domino occupies exactly two consecutive positions, and the goal is to compute the total number of distinct arrangements.

Linear Grid Case For a linear grid of m labeled positions, we aim to compute A_m^k, the number of ways to arrange k dominoes. Each domino occupies two positions, leaving $m - 2k$ empty spaces. These spaces are divided into $k + 1$ groups (before, between, and after the dominoes). The number of arrangements corresponds to the non-negative integer solutions of:

$$x_1 + x_2 + \cdots + x_{k+1} = m - 2k,$$

where $x_i \ge 0$ represents the empty positions in each segment. The solution is given by:

$$A_m^k = \binom{m-k}{k}.$$

Circular Grid Case For a circular arrangement, the positions form a closed loop. Each position can be:

- Occupied by a domino paired with the previous position.
- Occupied by a domino paired with the next position.
- Unoccupied.

This introduces an additional challenge since the circular nature prevents a direct division of the empty spaces into groups. To handle this, we fix one domino's position (breaking the symmetry) and analyze the remaining $m - 2$ positions linearly.

The number of ways to place k dominoes on a circular grid of m positions is:

$$W_m^k = 2A_{m-2}^{k-1} + A_{m-1}^k,$$

where A_{m-2}^{k-1} accounts for configurations where the fixed domino is paired with the next position, and A_{m-1}^k accounts for configurations where the fixed domino is paired with the previous position.

Substituting $A_m^k = \binom{m-k}{k}$, we have:

$$W_m^k = 2\binom{m-1-k}{k-1} + \binom{m-1-k}{k}.$$

Simplifying further:

$$W_m^k = \binom{m-1-k}{k-1} + \binom{m-k}{k}.$$

Using the combinatorial identity:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

we rewrite:

$$W_m^k = \left(\frac{k}{m-k} + 1\right)\binom{m-k}{k}.$$

Finally:

$$W_m^k = \frac{m}{m-k}\binom{m-k}{k}.$$

Thus, the number of ways to place k identical, non-overlapping dominoes on m circular positions is:

$$W_m^k = \frac{m}{m-k}\binom{m-k}{k}.$$

4.4.2 Solution of the Ménage Problem

Returning to the original formula:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

To arrange k couples around a circular table of size $2n$:

- Choose k couples from n, which can be done in $\binom{n}{k}$ ways.
- Arrange k couples as blocks (dominoes), multiplied by $k!$.
- Alternate starting with either a man or a woman (2 choices).
- Place the remaining $(n-k)$ men and $(n-k)$ women arbitrarily, yielding $((n-k)!)^2$ arrangements.

Thus:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = 2\binom{n}{k} k!((n-k)!)^2 W_{2n}^k.$$

Simplifying:

$$\begin{aligned}\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| &= 2(n!)(n-k)!W_{2n}^k,\\ &= 2(n!)(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k}.\end{aligned}$$

Finally, the solution to the Ménage problem using IEP is:

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= 2(n!)^2 - \sum_{k=1}^{n} (-1)^{k-1} 2(n!)(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k},\\ &= 2(n!)\sum_{k=0}^{n}(-1)^k(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k}.\end{aligned}$$

5. Bonferroni Inequalities and Inclusion-Exclusion Principle

Theorem 5.1 (Bonferroni Inequalities).
Let $A_1, A_2, \dots, A_n$ be n sets. The Bonferroni inequalities are:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \quad \text{(if } m \text{ is even)}$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \leq \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \quad \text{(if } m \text{ is odd.)}$$

Proof. Exercise. □