Chapter 1: Counting

1. Introduction

1.1 What is Discrete Mathematics?

Discrete Mathematics is a field of mathematics that deals with distinct and countable objects, rather than quantities that change continuously. It focuses on the study of structures that are fundamentally discrete, such as numbers, logical statements, and algebraic structures. The topics covered in this field form the backbone of areas like computer science, cryptography, and various branches of pure mathematics.

1.1.1 Key Concepts in Discrete Mathematics:

- **Sets:** Collections of distinct objects or elements.
- **Logic:** Principles of reasoning and constructing valid arguments.
- **Algebraic Structures:** The study of mathematical structures such as groups, rings, and fields, which are essential in various areas of mathematics and computer science.
- **Number Theory:** The study of properties and relationships of integers.
- **Combinatorics:** The art of counting, organizing, and selecting objects. In our course, we will focus on **enumerative combinatorics**, which is concerned with counting discrete structures and analyzing how they can be organized or constructed. Often in combinatorics, we are faced with answering key questions such as:
  - Is the configuration possible?
  - In how many ways can the configuration be made?
  - How do we go about finding such a configuration?

Examples.
Here are some example questions in combinatorics.

- How many distinct ways can the letters in the word "NHSM" be arranged?
- A class has 12 students, and 3 different prizes (gold, silver, bronze) are to be awarded. In how many ways can the prizes be distributed?
- How many ways can you arrange 6 balls into 4 boxes?
- How many ways can you assign 5 distinct colors to 5 identical chairs in a row?
- How many passwords of length n are possible using 26 letters?
- A company has 7 tasks to assign to 3 employees. In how many ways can the tasks be assigned if each employee must get at least one task?

Remark.
It is very important when doing the enumeration (counting the number of possible ways) to understand the distinction between listing all the configurations and determining their number.

1.2 Why Study Combinatorics?

Combinatorics is a fundamental area of Discrete Mathematics that provides tools to solve problems involving counting, organization, and construction of discrete structures. It plays a crucial role in many applications, especially in areas requiring detailed analysis of finite systems.

1.2.1 Applications of Combinatorics:

- **Computer Science:** Combinatorics is essential for designing efficient algorithms, data structures, and cryptographic systems, as well as in programming and network theory.
- **Engineering:** Combinatorics helps optimize system designs, network configurations, and resource allocations.

- **Mathematics:** It supports advancements in areas like number theory, algebraic structures, and discrete optimization.
- **Statistics:** Combinatorial methods lay the foundation for probability theory and statistical models.
- **Physics:** Applied in fields such as quantum mechanics and statistical physics, where discrete models and structures are essential.

Studying combinatorics equips you with powerful techniques for logical reasoning, problem-solving, and analyzing complex systems. These skills are highly valuable in technology, mathematics, engineering, and beyond.

Exercise.
How many different 2-letter words, including nonsense words can be produced by arranging the letters H,I,M?

Solution.
The set of all possible 2-letter words, including nonsense words is HI,HM,IM,IH,MH,MI,II,HH,MM, there is 9 such words.

Let change the question a little

Exercise.
How many different 26-letter words, including nonsense words can be produced by arranging the letters of the alphabet?

Solution.
There are 26^{26} such words, even using a computer that prints 1 billion words per second it will take 1.27×10^{17} centuries to print all the possible words.

2. Fundamental Principles of Counting: Product or Sum?

In this section, we introduce two fundamental counting principles: the *addition principle* and the *multiplication principle*, both of which are essential for solving combinatorial problems.

First, let's define the *Cardinality of a Set* which is the number of elements in a finite set S. The cardinality of S is denoted by $|S|$. For example, if $S = \{a, b, c, d\}$, then:

$$|S| = 4$$

Similarly, if $T = \{1, 2, 3, 4, 5, 6, 7\}$, then:

$$|T| = 7$$

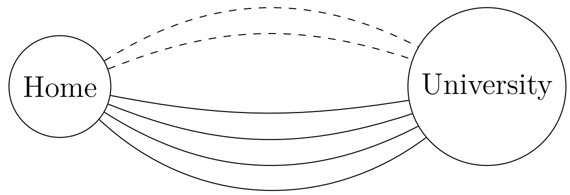
2.1 Addition principle

Theorem 2.1 (Addition Principle).
If A and B are two disjoint sets ($A \cap B = \emptyset$), then:

$$|A \cup B| = |A| + |B|$$

Examples.

1. Suppose you have 2 different train routes and 4 different bus routes from your home to the university. Let A be the set of train routes and B the set of bus routes. According to the Addition Principle, you have a total of $|A| + |B| = 6$ possible routes to reach the university.



2. In a class, we have 8 boys and 5 girls. We want to choose a representative of the group. By the Addition Principle, we have $|Boys| + |Girls| = 13$ possible choices .

Corollary 2.2 (Generalised Addition Principle).
If $A_1, A_2, \ldots, A_n$ are n pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for all $i \neq j$), then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

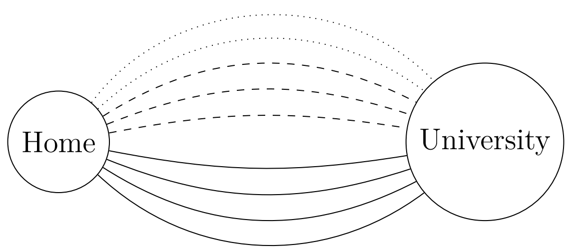
Example.
Suppose you have three different modes of transportation from your home to the university:

- 3 different train routes (A_1),
- 4 different bus routes (A_2),
- 2 different tram routes (A_3).

Since these sets of routes are pairwise disjoint (you can't take more than one mode of transportation at the same time), So by generalised adddition principle the total number of routes available is the sum of the routes for each mode of transportation:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 3 + 4 + 2 = 9$$

Thus, you have 9 total route options to get to the university.



Theorem 2.3 (Addition Principle: Alternate Form).
If a problem consists of solving one of k tasks, and each task i (where $1 \leq i \leq k$) can be done in n_i ways, then the total number of solutions to the problem is:

$$\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k.$$

2.2 Multiplication principle

Let us consider the set $A \times B$, which contains all ordered pairs (a, b) where $a \in A$ and $b \in B$. We call $A \times B$ the *Cartesian product* of A and B.

Theorem 2.4 (Multiplication Principle).
If A and B are two sets, then:

$$|A \times B| = |A| \cdot |B|$$

Examples. 1. Suppose you need to travel from your home to the university. You can:

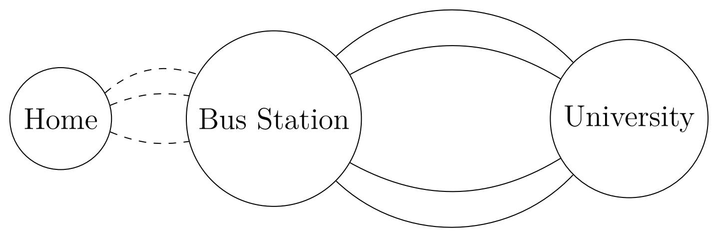
(a) Choose one of 3 train routes from home to the bus station.

(b) Choose one of 4 bus routes from the bus station to the university.

By the Multiplication Principle, the total number of ways to complete the journey is:

$$\text{Total routes} = 3\,(\text{train routes}) \times 4\,(\text{bus routes}) = 12\,\text{routes}.$$

Thus, there are 12 possible ways to travel from home to the university.



2. Suppose a class has 8 boys and 5 girls, and we need to form a committee consisting of one boy and one girl. The number of ways to choose a boy and a girl for the committee is determined by the Multiplication Principle:

(a) First, you choose one of the 8 boys.

(b) Then, you choose one of the 5 girls.

By the Multiplication Principle, the total number of ways to form the committee is:

$$\text{Total ways} = 8\,(\text{boys}) \times 5\,(\text{girls}) = 40\,\text{ways}.$$

Thus, there are 40 possible ways to form the committee.

Corollary 2.5 (Generalized Multiplication Principle).
If $A_1, A_2, \ldots, A_n$ are n sets, then:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$$

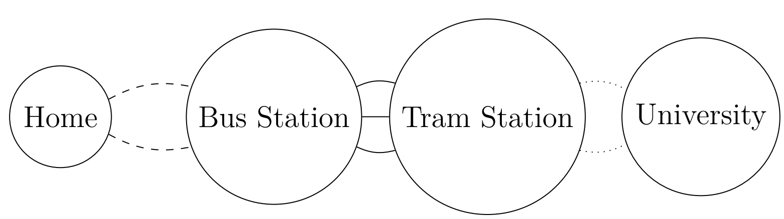
Example.
Suppose you want to travel from home to the university. The journey consists of three stages:

1. Choose 2 different train routes to go from home to the bus station,
2. Choose 3 different bus routes to go from the bus station to the tram station,
3. Choose 2 different tram routes to reach the university.

By the **Multiplication Principle**, the total number of ways to complete the journey is:

$$|A_1 \times A_2 \times A_3| = 2\,(\text{train routes}) \times 3\,(\text{bus routes}) \times 2\,(\text{tram routes}) = 12\,\text{ways}.$$

Thus, there are 12 different ways to travel from home to the university.



Theorem 2.6 (Multiplication Principle: Alternate Form).
If a problem consists of solving k tasks sequentially, and each task i (where $1 \le i \le k$) can be performed in n_i ways, then the total number of solutions to the problem is:

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k.$$

3. Distinguishability, Ordering, and Repetition

In combinatorial problems, we often need to select a subset A from the elements of a finite set S, based on certain criteria. To solve these problems, we must consider the following three key properties:

Definition 3.1 (Distinguishability).
Two elements of the set S are considered indistinguishable if they are identical with respect to the property in question. Otherwise, they are distinguishable if they differ with respect to this property.

Example.
In the following example, depending on the property we consider, the elements are distinguishable or indistinguishable:

- **By color**: We have 3 distinguishable groups, each containing 4 indistinguishable squares.
- **By number**: We have 4 distinguishable groups, each containing 3 indistinguishable squares.
- **By both color and number**: We have 12 pairwise distinguishable squares.

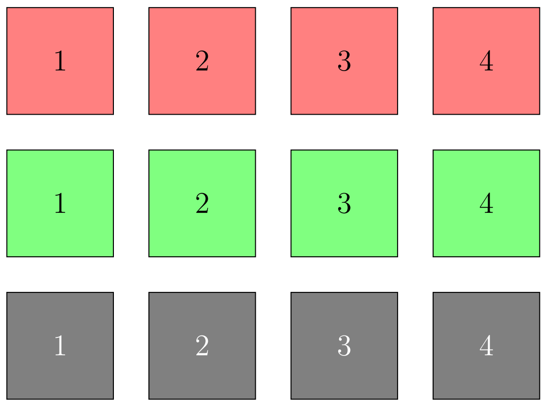


Figure 1: Distinguishability based on different properties

Definition 3.2 (Ordering).
If the elements in the selected subset A are treated equally and play the same role, the selection is unordered. If the elements can occupy different roles (i.e., their positions matter), the selection is ordered (a list).

Example.
Consider the 100-meter final at the Olympics with 8 participants:

- If we want to randomly select 3 runners, the selection is unordered.
- If we want to select 3 runners for gold, silver, and bronze medals, the selection is ordered since the positions matter.

Remark.
If the items are truly indistinguishable, then "order" becomes meaningless!

Definition 3.3 (Repetition).
If an element from the set S can appear more than once in the selection, repetition is allowed. If each element can only appear once, repetition is not allowed.

Example.
Suppose we need to select a leader from a group of students each day. Repetition is possible because a student can be chosen more than once as the leader. Does the order matter in this case?

Now, let's consider the possible cases of selecting a part of k elements under the conditions of ordering and repetition.

3.1 Arrangements (k-Permutations)

Definition 3.4.
Let S be a set of n elements $\{e_1, e_2, \ldots, e_n\}$. An **arrangement** (or **k-permutation**) of k elements from S refers to any ordered sequence of k elements $(a_1, a_2, \ldots, a_k)$ selected from S. In other words, an arrangement is a selection **with order** of k elements from a set S of n elements.

Examples.

- Let $S := \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The numbers $123, 112, 344, 789, 258$ are arrangements of 3 elements from S.
- Let $S := \{A, B, C, D, E, F\}$. The words ABC, DEF, AAB, CDC, CDA are arrangements of 3 elements from S.

Remark.
An arrangement can simply be described as a listing of elements.

3.1.1 k-permutations without repetition

Definition 3.5.
A **k-permutation without repetition** is an arrangement of k distinct elements selected from a set S, where no element can appear more than once. In other words, it is an ordered sequence of k different elements chosen from S.

Example.
From the previous examples of arrangements, $123, 789$ and 258 are 3-permutations with repetition from the first set, while ABC, DEF and CDA are 3-permutations with repetition from the second set.

Theorem 3.6 (Number of k-permutations without repetition)**.**
The number of all k-permutations without repetition from a set S of n elements, denoted by $(n)_k$, is given by:

$$(n)_k = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

Exercise.
- Let $S = \{A, B, C\}$. How many 2-permutations without repetition can be formed from the set S? List all the possible 2-permutations.
- Let $S = \{A, B, C, D, E, F\}$. How many 4-permutations without repetition can be formed from the set S?

Solution.
- The number of 2-permutations from $S = \{A, B, C\}$ is:
$$(3)_2 = 3 \times 2 = 6$$
The possible 2-permutations are: AB, AC, BA, BC, CA, CB.
- The number of 4-permutations from $S = \{A, B, C, D, E, F\}$ is:
$$(6)_4 = 6 \times 5 \times 4 \times 3 = 360$$
Thus, there are 360 possible 4-permutations.

3.1.2 k-permutations with repetition

Definition 3.7.
A **k-permutation with repetition** is an arrangement of k elements selected from a set S, where the repetition of elements is allowed. In other words, it is an ordered sequence of k elements, where each element is chosen from S and can appear more than once.

Example.
From the previous examples, 112 and 344 are 3-permutations with repetition from the first set, while AAB and CDC are 3-permutations with repetition from the second set.

Theorem 3.8 (Number of k-permutations with repetition)**.**
The number of all k-permutations with repetition from a set S of n elements, denoted by $\mathcal{P}(n, k)$, is given by:

$$\mathcal{P}(n, k) = n^k$$

Remark.
The repetition of elements here is allowed and unlimited.

Exercise.
Let $S = \{1, 2, 3\}$. How many 2-permutations with repetition can be formed from the set S? List all the possible 2-permutations.

Solution.
The number of 2-permutations with repetition from the set of $n = 3$ elements is given by:

$$\mathcal{P}(n, k) = n^k$$

Here, $n = 3$ and $k = 2$, so the total number of 2-permutations is:

$$\mathcal{P}(3, 2) = 3^2 = 9$$

Now, let's list all the possible 2-permutations with repetition:

$$11, 12, 13, 21, 22, 23, 31, 32, 33$$

Thus, there are 9 possible 2-permutations with repetition.

3.1.3 Permutations without Repetition

Before defining a permutation, let $n!$ represent the factorial of a non-negative integer n, which is defined as:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1,$$

with the special case $0! = 1$.

Definition 3.9.
A permutation of a set $S = \{e_1, e_2, \ldots, e_n\}$ is an arrangement of all the elements of S without repetition (i.e., an n-permutation without repetition).

Example.
Let $S = \{A, B, C\}$. The set S has 3 elements, and the possible permutations (arrangements without repetition) are:

$$ABC, ACB, BAC, BCA, CAB, CBA$$

Thus, there are 6 permutations of S.

Theorem 3.10 (Number of Permutations)**.**
The number of permutations of a set S with n elements, denoted by P_n, is given by:

$$P_n = n!$$

Exercise.
How many nonsense words of length 26 can be formed using all the letters of the English alphabet without repetition?

Solution.
Since the word must be 26 letters long and use all the letters of the English alphabet without repetition, this is a permutation of all 26 letters. The number of such permutations is denoted by

P_{26}, where:

$$P_{26} = 26!$$

Therefore, the total number of possible nonsense words is:

$$P_{26} = 26! = 26 \times 25 \times 24 \times \cdots \times 2 \times 1.$$

Proposition 3.11.

- For all $n \geq 0$,
$$P_n = (n)_n = n!$$
- More generally, for $k \leq n$, the number of k-permutations of a set with n elements is given by:
$$(n)_k = \frac{n!}{(n-k)!}$$
- Additionally, the relationship between permutations can be expressed as:
$$P_n = (n)_k P_{n-k}$$
where $(n)_k = n \times (n-1) \times \cdots \times (n-k+1)$ represents the number of ways to arrange k elements from n elements.

Exercise.
Prove the proposition.

3.2 Combinations, Binomial Theorem and Pascal's Triangle

Definition 3.12.
A combination is a selection of a subset from a finite set, where the order of selection does not matter. Simply put, a combination of a set is a subset of that set.

Examples.

- $\{1, 2, 4\}$ and $\{5, 4, 1\}$ are both valid combinations of size 3 from the set $\{1, 2, 3, 4, 5\}$.
- $\{A, B, C\}$ and $\{B, A, C\}$ represent the same combination from the set $\{A, B, C, D\}$, because the order does not matter in a combination.

3.2.1 k-combination without repetition

Definition 3.13.
A k-combination without repetition from a set S of n elements is a selection of k elements from S, where order does not matter. Simply put, it is a subset of k elements from S.

Examples.

- The 2-combinations of the set $\{A, B, C, D\}$ are as follows:
$$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}$$
Thus, there are 6 possible 2-combinations.
- The combinations of the set $S = \{1, 2, 3\}$ are as follows:
  - The 0-combination: $\{\}$ (the empty set)
  - The 1-combinations: $\{1\}, \{2\}, \{3\}$
  - The 2-combinations: $\{1, 2\}, \{1, 3\}, \{2, 3\}$
  - The 3-combination: $\{1, 2, 3\}$

  Thus, there are 8 possible combinations in total, which are all the subsets of S, including the empty set.

Theorem 3.14 (Number of k-combinations without repetition).
The number of all k-combinations without repetition from a set S of n elements, denoted by $\binom{n}{k}$, is given by:

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

The quantities $\binom{n}{k}$ are also called binomial coefficients.

Exercise.

- How many ways can we select a team of 3 people from a group of 3 persons $S = \{P1, P2, P3, P4, P5\}$? List all the possible teams.
- From a group of 10 people (6 boys and 4 girls), how many ways can we select a team containing:
  - 4 people?
  - 2 boys and 2 girls?

Solution.
- To select a team of three people from a group of five persons $S = \{P1, P2, P3, P4, P5\}$, we calculate the number of 3-combinations without repetition from 5 elements:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2!} = \frac{5 \times 4}{2!} = 10.$$

  The possible teams are:

$$\{P1, P2, P3\}, \{P1, P2, P4\}, \{P1, P2, P5\}, \{P1, P3, P4\}, \{P1, P3, P5\},$$
$$\{P1, P4, P5\}, \{P2, P3, P4\}, \{P2, P3, P5\}, \{P2, P4, P5\}, \{P3, P4, P5\}.$$

- From a group of 10 people (6 boys and 4 girls):
  - To select 4 people, we calculate the number of 4-combinations from 10:

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

    Therefore, there are 210 possible teams of 4 people.
  - To select 2 boys from 6 and 2 girls from 4, we calculate:

$$\binom{6}{2} \times \binom{4}{2} = \frac{6 \times 5}{2!} \times \frac{4 \times 3}{2!} = 15 \times 6 = 90.$$

    Therefore, there are 90 possible teams with 2 boys and 2 girls.

3.2.2 Binomial coefficients triangle (Pascal's triangle)

If we arrange the binomial coefficients $\binom{n}{k}$ in a triangle, where n represents the row number and k represents the position in the row (column number), we obtain what is called the Binomial Coefficient Triangle (or Pascal's Triangle). The following figure illustrates the triangle:

n/k	0	1	2	3	4	5	6	7	8	9
0	$\binom{0}{0}=1$									
1	$\binom{1}{0}=1$	$\binom{1}{1}=1$								
2	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$							
3	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{3}=1$						
4	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{4}=1$					
5	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$				
6	$\binom{6}{0}=1$	$\binom{6}{1}=6$	$\binom{6}{2}=15$	$\binom{6}{3}=20$	$\binom{6}{4}=15$	$\binom{6}{5}=6$	$\binom{6}{6}=1$			
7	$\binom{7}{0}=1$	$\binom{7}{1}=7$	$\binom{7}{2}=21$	$\binom{7}{3}=35$	$\binom{7}{4}=35$	$\binom{7}{5}=21$	$\binom{7}{6}=7$	$\binom{7}{7}=1$		
8	$\binom{8}{0}=1$	$\binom{8}{1}=8$	$\binom{8}{2}=28$	$\binom{8}{3}=56$	$\binom{8}{4}=70$	$\binom{8}{5}=56$	$\binom{8}{6}=28$	$\binom{8}{7}=8$	$\binom{8}{8}=1$	
9	$\binom{9}{0}=1$	$\binom{9}{1}=9$	$\binom{9}{2}=36$	$\binom{9}{3}=84$	$\binom{9}{4}=126$	$\binom{9}{5}=126$	$\binom{9}{6}=84$	$\binom{9}{7}=36$	$\binom{9}{8}=9$	$\binom{9}{9}=1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Binomial coefficients triangle

The first property to note is that

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1.$$

Combinatorially, this can be interpreted as follows: the number of ways to select 0 elements from a set of n elements is 1 (the empty set), and the number of ways to select all n elements from the set is also 1 (the entire set).

Theorem 3.15 (Symmetry Property)**.**
For any integers n and k such that $0 \leq k \leq n$, the following holds:

$$\binom{n}{k} = \binom{n}{n-k}$$

Theorem 3.16 (Recurrence Relation)**.**
For any integers n and k, $(0 \leq k \leq n)$, the binomial coefficient satisfies the following recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Remark.
For now, we assume that if $k > n$, $k < 0$, or $n < 0$, the binomial coefficient $\binom{n}{k} = 0$.

Theorem 3.17 (Vandermonde's Identity)**.**
For any non-negative integers m, n and k, Vandermonde's identity is given by:

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}$$

Exercise.
Prove the following properties of binomial coefficients using both algebraic and combinatorial argu-

ments:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \quad \text{for } k \geq 1,$$
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

3.2.3 The Binomial Theorem

One of the most widely used theorems in mathematics is the expansion of $(a+b)^n$, known as the Binomial Theorem.

Theorem 3.18 (The Binomial Theorem)**.**
For any variables a and b, and for each positive integer n, the expansion of $(a+b)^n$ is:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

Exercise.
Determine the coefficient of $a^{17}b^9$ in the expansion of $(a+b)^{26}$.

Solution.
The general term in the expansion of $(a+b)^{26}$ is given by $\binom{26}{k}a^{26-k}b^k$.
To find the term $a^{17}b^9$, we need $26-k=17$, which gives $k=9$.
Therefore, the coefficient is:

$$\binom{26}{9}.$$

3.3 Permutations with (limited) repetitions, Multinomial Coefficients, and the Multinomial Theorem

3.3.1 The MISSISSIPPI Problem

Consider the following problem: How many distinct words can be formed by rearranging the letters of the word "MISSISSIPPI"?

At first glance, one might assume there are 11! possible arrangements since the word consists of 11 letters. However, this is incorrect because permuting identical letters yields the same word multiple times. To solve this problem, we must account for the repeated letters by selecting the positions for each without permuting them internally. Here's how it works:

- $\binom{11}{1}$ possible positions for the 'M',
- $\binom{11-1}{4}$ positions for the 'I',
- $\binom{11-1-4}{4}$ positions for the 'S',
- $\binom{11-1-4-4}{2}$ positions for the 'P'.

Using the multiplication principle, the total number of distinct permutations is given by:

$$\binom{11}{1}\binom{11-1}{4}\binom{11-1-4}{4}\binom{11-1-4-4}{2} = \frac{11!}{1!10!}\frac{10!}{4!6!}\frac{6!}{4!2!}\frac{2!}{2!} = \frac{11!}{1!4!4!2!}$$

This formula represents the total number of ways to permute all 11 letters, while dividing by the factorial of repeated letters to exclude identical permutations.

3.3.2 Permutations with (limited) repetitions

More generally, we can express this concept in the form of the following theorem:

Theorem 3.19 (Permutation with Repetition)**.**
Let S be a collection of k distinct elements $e_1, e_2, \ldots, e_k$, where each element e_i is repeated n_i times

$(1 \le i \le k)$, and the total number of elements with repetitions is n (i.e., $n_1 + n_2 + \cdots + n_k = n$). The number of permutations of all the elements in S, denoted by $\binom{n}{n_1, n_2, \ldots, n_k}$, is given by:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-\sum_{i=1}^{k-1} n_i}{n_k} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

where the coefficients $\binom{n}{n_1, n_2, \ldots, n_k}$ are called multinomial coefficients.

Remarks.

- The collection S can be denoted as:
$$\{n_1 \cdot e_1, n_2 \cdot e_2, \ldots, n_k \cdot e_k\},$$
which is called a *multiset*. Unlike sets, multisets can contain the same element multiple times.
- Infinite repetition is represented by ∞. For example, the multiset:
$$\{\infty \cdot a, 5 \cdot b, 3 \cdot c\}$$
means that element a appears infinitely often, b appears 5 times, and c appears 3 times.
- Problems involving infinite repetition are generally simpler than those with finite repetition. Given a multiset:
$$A = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\},$$
the task is to determine how many permutation of n elements can be formed. Since each element can be used repeatedly without limit, the number of such arrangements is:
$$n^n.$$

3.3.3 Properties of Multinomial Coefficients

Multinomial coefficients generalize binomial coefficients by representing the number of ways to divide n distinct objects into k groups of sizes $n_1, n_2, \ldots, n_k$, where the total size of all groups equals n.

Theorem 3.20 (Recursive Formula for Multinomial Coefficients).
Let n be a positive integer, and let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n = n_1 + n_2 + \cdots + n_k$ with $k \ge 2$. The multinomial coefficient can be expressed recursively as follows:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \sum_{i=1}^{k} \binom{n-1}{n_1, \ldots, n_i - 1, \ldots, n_k} \\ &= \binom{n-1}{n_1 - 1, n_2, \ldots, n_k} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_k} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_k - 1}.\end{aligned}$$

Exercise.
Prove the theorem both algebraically and combinatorially.

3.3.4 Multinomial Theorem

The multinomial theorem extends the binomial theorem.

Theorem 3.21 (Multinomial Theorem).
For any positive integer n and any k-tuple of non-negative integers $x_1, x_2, \ldots, x_k$, the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is given by:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1 + n_2 + \cdots + n_k = n \\ n_1 \ge 0, \ldots, n_k \ge 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $\binom{n}{n_1, n_2, \ldots, n_k}$ are the multinomial coefficients.

Exercise.

- Provide a combinatorial proof using similar arguments to those used in the binomial theorem.
- Prove the multinomial theorem using induction:
  - On n.
  - On k.

Example.
Expand $(x_1 + x_2 + x_3)^4$:

$$(x_1 + x_2 + x_3)^3 = \sum_{\substack{n_1+n_2+n_3=3 \\ n_1,n_2,n_3 \geq 0}} \binom{3}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3},$$
$$= \binom{3}{0,0,3} x_3^3 + \binom{3}{0,3,0} x_2^3 + \binom{3}{0,1,2} x_2 x_3^2 + \binom{3}{0,2,1} x_2^2 x_3^1 + \binom{3}{1,1,1} x_1 x_2 x_3$$
$$+ \binom{3}{1,2,0} x_1 x_2^2 + \binom{3}{1,0,2} x_1 x_3^2 + \binom{3}{2,0,1} x_1^2 x_3 + \binom{3}{2,1,0} x_1^2 x_2 + \binom{3}{3,0,0} x_1^3.$$

Exercise.
Expand $(a + b + 2c)^5$ and find the coefficients of:

- a^2b^2c
- a^3bc
- b^5

3.4 Combinations with repetitions

We have already seen the k-combinations without repetition, which are counted by $\binom{n}{k}$. Now, let us consider the following problem: let $S = \{1, 2, 3\}$. What are the 2-combinations of S without repetition? Clearly, they are

$$\{1, 2\}, \{1, 3\}, \{2, 3\}.$$

Now, let us consider the same problem, but with repetition allowed. The solutions are:

$$\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}$$

. Thus, we have 6 solutions.

Can you now answer the question: if the set is $S = \{1, 2, 3, \ldots, n\}$ and we want to form a k-combination with repetition allowed, how many such combinations are there?

Definition 3.22.
A *k-combination with repetition* from a set $S = \{1, 2, 3, \ldots, n\}$ is an unordered selection of k elements, where repetition of elements is allowed.

Remarks.

- It is evident that the repetitions of elements cannot exceed k. If $x_1, x_2, \ldots, x_n$ represent the repetitions of the elements $e_1, e_2, \ldots, e_n$, then we must have the equation $x_1 + x_2 + \cdots + x_n = k$ with $0 \leq x_i \leq k$ for each $1 \leq i \leq n$.
- The problem of finding a k-combination with repetition is equivalent to forming a multiset of k elements, denoted as $\{x_1 \cdot e_1, x_2 \cdot e_2, \ldots, x_n \cdot e_n\}$, where the sum $x_1 + x_2 + \cdots + x_n = k$ holds.

Theorem 3.23 (Number of k-combinations with repetition)**.**
The number of k-combinations with repetition allowed from a set $S = \{1, 2, 3, \ldots, n\}$, denoted $\left(\!\binom{n}{k}\!\right)$,

is given by:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

3.4.1 Application: Number of Integer Solutions of Simple Linear Equations

An important application of k-combinations with repetition is counting the number of nonnegative integer solutions to linear equations. Consider the following general problem:

Question: How many nonnegative integer solutions exist for the equation:

$$x_1 + x_2 + \cdots + x_n = k?$$

Theorem 3.24.
The number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = k$ is given by:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{k+n-1}{k}.$$

Exercise.
Solve the following problems:

- How many nonnegative integer solutions are there for the equation:

$$x_1 + x_2 + x_3 = 6?$$

- How many solutions exist for the equation:

$$x_1 + x_2 + x_3 + x_4 = 18,$$

  where $x_1 \geq 3$, $x_2 \geq 5$, $x_3 \geq 1$, and $x_4 \geq 0$?

Exercise.
How many nonnegative integer solutions exist for the inequality:

$$x_1 + x_2 + \cdots + x_n \leq k?$$

Appendix A: History of the Binomial Coefficients Triangle (BCT)

- **Earliest Mentions:** The Binomial Coefficients Triangle (BCT) dates back to Pingala (2nd century B.C.), though only fragments of his work survived. Varãhamihira (505 A.D.) later described the additive rule.
- **Islamic Contributions:**
  - **Al-Khalil ibn Ahmad Al-Farahidi (718-786):** In his book *Kitab Al-'Ayn*, Al-Khalil used binomial coefficients to enumerate combinations of letters in the Arabic alphabet.
  - **Al-Kharji (953-1029):** He proved the recurrence relation and the binomial expansion formula.
  - **Omar Alkhayam (1048-1131)** and **Al-Samw'al (1200s):** They further developed the BCT. Al-Samw'al's manuscript, *Al-Bahir fi Al-Hissab*, contains an early depiction of the triangle.
  - **Al-Tusi (1201-1273):** He established key fundamental relations of the BCT.
- **Chinese Contributions:** In the 11th century, the Chinese mathematician Yang Hui (1238-1298) documented the BCT, preserved in his works.
- **European Contributions:**
  - The BCT appeared in Jordanus de Nemore's work in the 13th century.
  - **Pertrus Apianus (1495-1552):** Published the full triangle.
  - **Tartaglia (1500-1577):** Published the triangle in his book in 1556.
  - **Blaise Pascal (1623-1662):** Popularized the triangle in the 17th century in his work *Traité du triangle arithmétique*, and it became widely known as Pascal's Triangle.