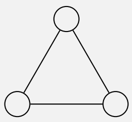
Worksheet 3: Introduction to graph theory

Exercise 1:

For each of the sequences, decide if it represents the degree sequence of a simple graph.

Answers to Exercise 1:

1. **1, 2, 3, 3, 5** No. The sum of the degrees is 14, which is even, but the highest degree (5) is greater than the number of vertices minus one (4). This violates the Handshaking Lemma.
2. **2, 3, 4, 5, 3, 4** No. The sum of the degrees is 21, which is odd. This violates the Handshaking Lemma.
3. **2, 2, 2** Yes. The sum of the degrees is 6, which is even, and it is possible to construct a simple graph with this degree sequence (e.g., a triangle).



4. **5, 3, 5, 1, 2, 2** No, We have two vertices of degree 5 so they are adjacents to all the others, therefoe there is no vertex with degree 1, at least 2.

Exercise 2:

Show that the number of K_3 subgraphs in a complete graph K_n with $n \geq 3$ is equal to $\frac{n(n-1)(n-2)}{6}$.

Answers to Exercise 2:

A K_3 subgraph is a triangle, which is formed by selecting 3 vertices out of n. The number of ways to choose 3 vertices from n is given by the combination formula:

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

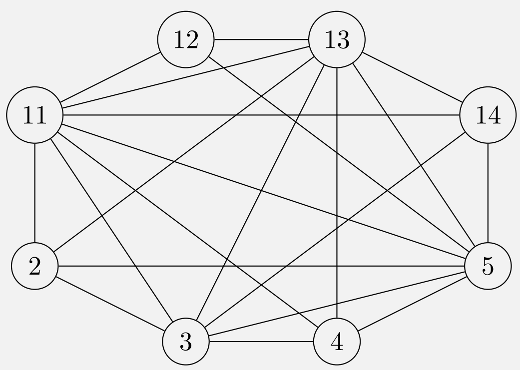
Therefore, the number of K_3 subgraphs in K_n is $\frac{n(n-1)(n-2)}{6}$.

Exercise 3:

Let G be a graph with vertices $V = \{2, 3, 4, 5, 11, 12, 13, 14\}$, and two vertices u and v are adjacent if $\gcd(u, v) = 1$.

Answers to Exercise 3:

1. **Draw G:** The graph G will have edges between vertices that are coprime. Below is a representation of the graph:



2. **Find $e(G), \delta(G), \Delta(G)$, $N(4)$, $N[3]$, and $d(13)$:**
   - $e(G) = \frac{1}{2}\sum_{v \in V} deg(v) = 21$.

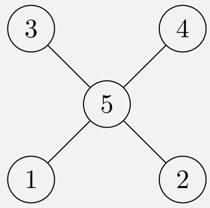
- $\delta(G) = 3$.
- $\Delta(G) = 7$.
- $N(4) = \{3, 5, 11, 13\}$.
- $N[3] = \{2, 3, 4, 5, 11, 13, 14\}$.
- $d(13) = 7$.

Exercise 4:

Does there exist a simple graph of order 5, such that there is a vertex of degree $\Delta(G) = 4$ and a vertex of degree $\delta(G) = 1$? What about two vertices of degree $\Delta(G) = 4$ and a vertex with degree $\delta(G) = 1$?

Answers to Exercise 4:

1. **Yes**, such a graph exists. For example, consider a star graph with one central vertex connected to 4 other vertices. The central vertex has degree 4, and the other vertices have degree 1. Below is the graph:



2. **No**, such a graph does not exist. If there are two vertices of degree 4 in a graph of order 5, they must be connected to all other vertices, including each other. This would leave no vertex with degree 1.

Exercise 17:

Suppose that G is a k-regular simple graph of order n and size m. Find a relation between k, n, and m.

Answers to Exercise 17:

In a k-regular graph, every vertex has degree k. By the Handshaking Lemma, the sum of the degrees of all vertices is equal to twice the number of edges. Therefore:

$$\sum_{v \in V} \deg(v) = 2m.$$

Since G is k-regular, the sum of the degrees is $n \cdot k$. Thus:

$$n \cdot k = 2m.$$

The relation between k, n, and m is:

$$m = \frac{n \cdot k}{2}.$$

Exercise 18:

In any graph with no K_3 (triangle), show that $n \geq \delta(G) + \Delta(G)$, where n is the order of the graph, $\delta(G)$ is the minimum degree, and $\Delta(G)$ is the maximum degree.

Answers to Exercise 18:

Let x be a vertex with degree $\Delta(G)$, and let y be a neighbor of x. Since G has no triangles, $N(x) - \{y\}$ and $N(y) - \{x\}$ are disjoint. Thus:

$$|N[x] \cup N[y]| = \Delta(G) + 1 + d(y) - 1 = \Delta(G) + d(y).$$

Because $|N[x] \cup N[y]| \leq n$, we have:

$$\Delta(G) + d(y) \leq n.$$

Since $d(y) \geq \delta(G)$, it follows that:

$$n \geq \Delta(G) + \delta(G).$$

Thus, $n \geq \Delta(G) + \delta(G)$ holds for any graph with no K_3.

Exercise 5:

Let G be a simple graph. Show that if $P = v_1 \dots v_l$ is a longest path in G, then all the neighbors of v_1 are in $\{v_2, \dots, v_l\}$.

Answers to Exercise 5:

Suppose there exists a neighbor u of v_1 that is not in $\{v_2, \dots, v_l\}$. Then we could extend the path P by adding u at the beginning, resulting in a longer path $u, v_1, \dots, v_l$. This contradicts the assumption that P is the longest path. Therefore, all neighbors of v_1 must be in $\{v_2, \dots, v_l\}$.