Worksheet 2: Inclusion-Exclusion principle

Answers to Exercise 1:

We divide the set of 100 integers into 50 pairs of consecutive integers, treating each pair as a pigeonhole

$$\{(1, 2), (3, 4), (5, 6), \ldots, (99, 100)\}.$$

Let the 51 chosen integers $\{i_1, i_2, \ldots, i_{50}\}$ represent the pigeons.

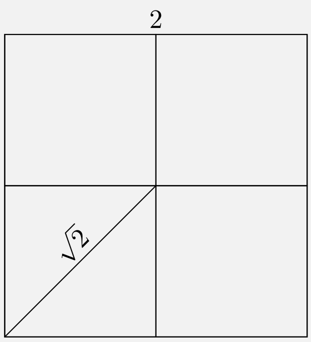
According to the Pigeonhole Principle (PP),

- Pigeons: The 51 chosen numbers from the integers 1 to 100.
- Pigeonholes: The pairs of consecutive integers $(1, 2), (3, 4), \ldots, (99, 100)$.

when we select 51 integers, at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Consequently, there exist 2 consecutive integers within the set of 51 chosen integers.

Answers to Exercise 2:

We partition the square with a side length of 2 into four smaller squares, each with a side length of 1, achieved by bisecting its sides.



Each of the five points must fall within one of these four small squares.

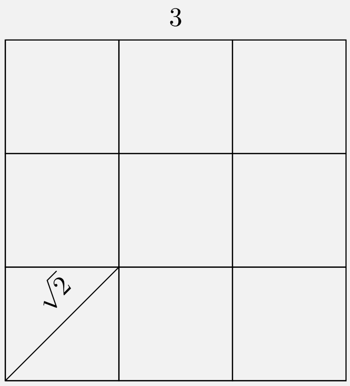
According to the Pigeonhole Principle:

- Pigeons: The 5 points in the square.
- Pigeonholes: 4 small squares of side 1.

there must exist a small square containing at least two of the five points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 3:

We partition the square with a side length of 3 into nine smaller squares, each with a side length of 1, achieved by trisecting its sides.



Each of the ten points must fall within one of these nine small squares.

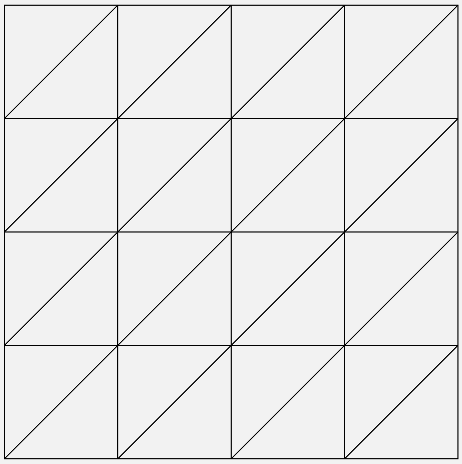
According to the Pigeonhole Principle:

- Pigeons: The 10 points in the square.
- Pigeonholes: 9 small squares of side 1.

There must exist a small square containing at least two of the ten points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 4:

Let us cut the square of side 1 into 32 congruent parts. This yields 32 parts of area $\frac{1}{32}$ each.



According to the generalized version of the Pigeonhole Principle,

- Pigeons: The 65 points inside the square.
- Pigeonholes: The 32 smaller parts obtained by dividing the larger square.

at least one of these 32 parts must contain at least three of our points.

Consequently, the triangle formed by the 3 points has an area less than $\frac{1}{32}$ (area of each small part).

Answers to Exercise 5:

We partition the set of 20 integers into 10 pairs, each with a sum of 21, treating each pair as a pigeonhole:

$$\{(1, 20), (2, 19), (3, 18), \ldots, (10, 11)\}$$

Let $\{i_1, i_2, \ldots, i_{11}\}$ represent the 11 chosen integers as pigeons.

According to the Pigeonhole Principle (PP),

- Pigeons: The 11 chosen numbers from the integers 1 to 20.
- Pigeonholes: The pairs of integers $\{(1, 20), (2, 19), (3, 18), \ldots, (10, 11)\}$ with a sum of 21.

When we select 11 integers, PP guarantees that at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Therefore, there exist 2 integers with a sum of 21.

Answers to Exercise 6:

- **Case 1:** Suppose every member of the group has at least one friend.

  In this scenario, each of the n members will have between 1 and $n-1$ friends. We can conceptualize these friend counts as holes, labeled with numbers from 1 to $n-1$, and the n members as pigeons.

  With $n-1$ holes and n pigeons, by the Pigeonhole Principle, there must be at least one hole that contains more than one pigeon. This implies that there exists a number from 1 to $n-1$ representing the count of friends, which is shared by more than one member.

  Thus, in a group of n members, there must be at least two individuals with an equal number of friends.

- **Case 2:** Assume there is a person in the group with no friends.

  In this case, excluding the person with no friends, consider the remaining $n-1$ members. If any one of them has zero friends, we immediately find two individuals with an identical number of friends.

  If none of the remaining $n-1$ members has zero friends, excluding the person with no friends, the remaining $n-1$ individuals will have between 1 and $n-2$ friends. This situation aligns with Case 1, and thus, even in this scenario, we would have two individuals with an identical number of friends.

Consequently, in a group of n people, there must always exist two persons with the same number of friends.

Answers to Exercise 7:

Consider the residues of integers when divided by 10.

There are 10 possible residues when an integer is divided by 10: $\{0, 1, 2, \ldots, 9\}$.

Now, let's divide the set of residues into pairs with a sum of 10 as follows:

$$(0), (1, 9), (2, 8), (3, 7), (4, 6), (5).$$

By the Pigeonhole Principle (PP), if we distribute 7 numbers among the 6 subsets, there must be at least two integers x and y in the same set, so either $r_x = r_y$ or $r_x + r_y = 10$.

- If $r_x = r_y$, then $x - y$ is divisible by 10.
- If $r_x + r_y = 10$, then $x + y$ is divisible by 10.

Therefore, in any set of 7 distinct integers, there must be two integers x and y such that either $x + y$ or $x - y$ is divisible by 10.

Answers to Exercise 8:

Let $x_1, x_2, \ldots, x_{15}$ represent the number of games played on each of the 15 days. Given that the team plays at least one game daily, we have $x_i \geq 1$ for all i.

The total number of games played in the 15-day season is expressed as the sum:

$$x_1 + x_2 + \ldots + x_{15} = 20$$

Now, let's define the cumulative sums:

$$S_1 = x_1$$
$$S_2 = x_1 + x_2$$
$$\ldots$$
$$S_{15} = x_1 + x_2 + \ldots + x_{15}$$

The objective is to find indices i and j such that $S_j - S_i = x_{i+1} + x_{i+2} + \ldots + x_j = 9$, and $j > i$. This ensures that during the consecutive days from the $(i+1)$-th day to the j-th day, the team played exactly 9 games.

Alternatively, we can find S_j such that $S_j = S_i + 9$. Consider the possible sums $T_i = S_i + 9$:

$$T_1 = S_1 + 9$$
$$T_2 = S_2 + 9$$
$$\ldots$$
$$T_{15} = S_{15} + 9$$

There are 15 different values for S_i ranging from 1 to 20, and 15 different values for T_i ranging from 10 to 29. By the Pigeonhole Principle, since we have 30 sums (S_j and T_i) (pigeons) distributed over the range $\{1, 2, \ldots, 29\}$ (pigeonholes), there must be two sums that are equal.

Hence, there exist i and j such that $S_j = T_i = S_i + 9$, and consequently, $S_j - S_i = x_{i+1} + x_{i+2} + \ldots + x_j = 9$.

This resolves the question.

Answers to Exercise 9:

- We start by dividing the set of $2n$ integers into n pairs, each with a sum of $2n + 1$, treating each pair as a pigeonhole:

  $$\{(1, 2n), (2, 2n - 1), (3, 2n - 2), \ldots, (n, n + 1)\}$$

  Let $\{i_1, i_2, \ldots, i_{n+1}\}$ denote the $n + 1$ chosen integers as pigeons. Applying the Pigeonhole Principle (PP):

  - **Pigeons:** The $n + 1$ chosen numbers from the integers 1 to $2n$.
  - **Pigeonholes:** The pairs of integers $\{(1, 2n), (2, 2n-1), (3, 2n-2), \ldots, (n, n+1)\}$ with a sum of $2n+1$.

  When $n + 1$ integers are selected, PP ensures that at least one pigeonhole (representing a pair of integers) must contain 2 pigeons (chosen integers). Consequently, there exist 2 integers with a sum of $2n + 1$.

- Similarly, if we partition the set into n pairs, each with a difference of n, treating each as a pigeonhole:

  $$\{(2n, n), (2n - 1, n - 1), (2n - 2, n - 2), \ldots, (n + 1, 1)\}$$

  According to the Pigeonhole Principle (PP):

  - **Pigeons:** The $n + 1$ chosen numbers from the integers 1 to $2n$.
  - **Pigeonholes:** The pairs of integers $\{(2n, n), (2n - 1, n - 1), (2n - 2, n - 2), \ldots, (n + 1, 1)\}$ with a difference of n.

  When $n + 1$ integers are selected, PP guarantees that at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Hence, there exist 2 integers with a difference of n.

Answers to Exercise 10:

Let the set of n integers be $\{a_1, a_2, \ldots, a_n\}$. Consider the cumulative sums of subsets from the set of n integers:

$$\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \ldots + a_n\}$$

There are n distinct values for these cumulative sums.

Now, in two cases:

Case 1: If any cumulative sum is divisible by n, we've found a subset with a sum divisible by n, solving the problem.

Case 2: If none of the cumulative sums is divisible by n, each sum has a remainder from 1 to $n-1$ when divided by n. With n cumulative sums and $n-1$ possible remainders, the Pigeonhole Principle implies at least two sums share the same remainder. Let $a_1 + a_2 + \ldots + a_i \equiv a_1 + a_2 + \ldots + a_j \pmod{n}$, where $i < j$.

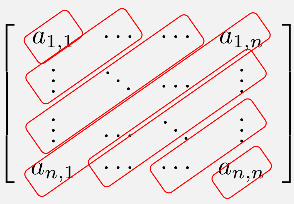
Then,

$$(a_1 + a_2 + \ldots + a_j) - (a_1 + a_2 + \ldots + a_i) = a_{i+1} + a_{i+2} + \ldots + a_j \equiv 0 \pmod{n}$$

This demonstrates that we've found a subset of consecutive integers with a sum divisible by n.

Answers to Exercise 11:

We examine the set of entries in the matrix A that form anti-diagonals, defined as the intersection of A with a line of slope 1. There are $2n-1$ such anti-diagonals in A, and each entry belongs to exactly one of them.



If two '1's are in the same anti-diagonal, they satisfy the property of being "one strictly above and strictly to the right of the other."

Since there are at least $2n$ entries equal to 1, and there are only $2n-1$ anti-diagonals in a matrix of size $n \times n$, by the Pigeonhole Principle, at least one anti-diagonal must contain at least two entries equal to 1. This is because we have more "pigeons" (entries equal to 1) than "pigeonholes" (anti-diagonals) to place them into.

Thus, there exist two entries in A with the property that one is strictly above and strictly to the right of the other. The statement is, therefore, proven.

Answers to Exercise 12:

To prove that the product is even, we just need to show that there is at least one factor that is even. Let's call this factor $(a_{i_k} - a_k)$.

Now, a number like $(a_{i_k} - a_k)$ is even only if both a_{i_k} and a_k are either both even or both odd. We say they have the same "parity" in this case.

So, the key is to consider the parity (even or odd) of the 5 numbers in the set A. To do this, let's separate them into two groups: one for even numbers and one for odd numbers. Since there are 5 numbers in total, by the Pigeonhole Principle (PP), at least 3 of them must be in the same group. Let's call these three elements a_1, a_2, and a_3.

Now, notice that the three elements we selected, $\{a_{i_1}, a_{i_2}, a_{i_3}\}$, are distinct from the original set $\{a_1, a_2, a_3\}$. So, without loss of generality, we can assume that $a_1 = a_{i_3}$.

This means that $a_{i_3} - a_3 = a_1 - a_3$. The difference on the right side is even because a_1 and a_3 have the same parity. Therefore, the factor $(a_{i_3} - a_3)$ is even, and this completes the proof.

Answers to Exercise 13:

Consider pairs of subjects (A, B) and treat each pair as a pigeonhole. If a student studies both subjects in a pair, we place a pigeon into that specific pigeonhole. Since a student chooses 4 subjects, the number of ways to choose 2 out of those 4 subjects is given by $\binom{4}{2}$, which is 6.

With 15 subjects, we have $\binom{15}{2} = 105$ pigeonholes (pairs of subjects). Therefore, each student contributes 6 pigeons, and considering 18 students, the total number of pigeons is $18 \times 6 = 108$.

According to the Pigeonhole Principle (PP), when the number of pigeons exceeds the number of pigeonholes, there must be at least two pigeons in the same pigeonhole.

Hence, by the Pigeonhole Principle, we can conclude that there are at least two students who have chosen the same pair of subjects.

Answers to Exercise 14:

For any integer $n \geq 1$, it can be uniquely expressed in the form $n = 2^a b$, where b denotes its odd factor. Applying this concept, consider the set $\{1, 2, 3, \ldots, 2n\}$ and replace each element with its odd factor. As there are only n odd numbers less than $2n$, there are precisely n choices for this factor. Consequently, according to the pigeonhole principle, when $n + 1$ numbers are selected from the set, at least two of them must share the same odd factor. These two numbers can be represented as $2^a b$ and $2^c b$, where it is evident that the smaller one divides the larger one.

Answers to Exercise 15:

Analyze the extended diagonals identified from 1 to 10 in the 10x10 chessboard arrangement presented below:

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	1
3	4	5	6	7	8	9	10	1	2
4	5	6	7	8	9	10	1	2	3
5	6	7	8	9	10	1	2	3	4
6	7	8	9	10	1	2	3	4	5
7	8	9	10	1	2	3	4	5	6
8	9	10	1	2	3	4	5	6	7
9	10	1	2	3	4	5	6	7	8
10	1	2	3	4	5	6	7	8	9

There are a total of 10 such diagonals, each comprising 10 squares. According to the pigeonhole principle, if 41 rooks are distributed on the chessboard, at least one diagonal must contain a minimum of five rooks, indicating the presence of 5 non-attacking rooks. We can see the problem as follow also.

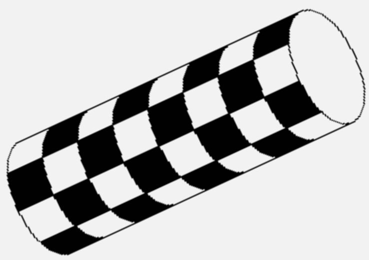


Figure 1: cylindrical chessboard

Answers to Exercise 18:

We transform the variables to simplify the constraints:

$$y_1 = x_1 - 1, \quad y_2 = x_2 + 2, \quad y_3 = x_3, \quad y_4 = x_4 - 3.$$

The equation becomes:

$$y_1 + y_2 + y_3 + y_4 = 16,$$

with constraints:

$$0 \leq y_1 \leq 4, \quad 0 \leq y_2 \leq 6, \quad 0 \leq y_3 \leq 5, \quad 0 \leq y_4 \leq 6.$$

We seek to find:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|,$$

where $\overline{A_i}$ denotes the subset of S satisfying the original constraints:

$$\overline{A_1} : 0 \leq y_1 \leq 4, \quad \overline{A_2} : 0 \leq y_2 \leq 6, \quad \overline{A_3} : 0 \leq y_3 \leq 5, \quad \overline{A_4} : 0 \leq y_4 \leq 6.$$

Using the inclusion-exclusion principle (IEP), we have:

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = &|S| - (|A_1| + |A_2| + |A_3| + |A_4|) \\ &+ (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &- (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &+ |A_1 \cap A_2 \cap A_3 \cap A_4|. \end{aligned}$$

Let A_i denote the subset of S violating the constraints:

$$A_1 : y_1 \geq 5, \quad A_2 : y_2 \geq 7, \quad A_3 : y_3 \geq 6, \quad A_4 : y_4 \geq 7.$$

The total number of nonnegative integer solutions without constraints is:

$$|S| = \binom{16+4-1}{16} = \binom{19}{16} = 969.$$

We calculate:

$$|A_1| = \binom{14}{11} = 364, \quad |A_2| = \binom{12}{9} = 220,$$

$$|A_3| = \binom{13}{10} = 286, \quad |A_4| = \binom{12}{9} = 220.$$

Next, we compute the pairwise intersections:

$$|A_1 \cap A_2| = \binom{7}{4} = 35, \quad |A_1 \cap A_3| = \binom{8}{5} = 56,$$

$$|A_1 \cap A_4| = \binom{7}{4} = 35, \quad |A_2 \cap A_3| = \binom{6}{3} = 20,$$

$$|A_2 \cap A_4| = \binom{5}{2} = 10, \quad |A_3 \cap A_4| = \binom{6}{3} = 20.$$

For the intersections of three or more subsets, we observe that:

$$|A_1 \cap A_2 \cap A_3| = 0, \quad |A_1 \cap A_2 \cap A_4| = 0,$$

$$|A_1 \cap A_3 \cap A_4| = 0, \quad |A_2 \cap A_3 \cap A_4| = 0,$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

Substituting the values into the IEP formula:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 969 - (364 + 220 + 286 + 220) + (35 + 56 + 35 + 20 + 10 + 20) - 0 + 0 = 55.$$

Answers to Exercise 19:

A surjection is a map $f : \mathbb{N}_n \to \mathbb{N}_k$ such that every element of $\mathbb{N}_k$ is mapped to by at least one element of $\mathbb{N}_n$. We seek to find the number of such surjections.

Let:

- S be the set of all maps from $\mathbb{N}_n$ to $\mathbb{N}_k$.
- For each $i \in \{1, 2, \ldots, k\}$, let A_i be the set of maps where the element y_i is **not** mapped to by any element of $\mathbb{N}_n$.

We seek to find:

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}| &= |S| - \sum_{m=1}^{k} (-1)^{m-1} \sum_{1 \le i_1 < \cdots < i_m \le k} |A_{i_1} \cap \cdots \cap A_{i_m}| \\ &= |S| - \sum_{i_1=1}^{k} |A_{i_1}| + \sum_{1 \le i_1 < i_2 \le k} |A_{i_1} \cap A_{i_2}| - \cdots + (-1)^k |A_1 \cap A_2 \cap \cdots \cap A_k|.\end{aligned}$$

which represents the number of maps where every element of $\mathbb{N}_k$ is mapped to by at least one element of $\mathbb{N}_n$ (i.e., the number of surjections).

The total number of maps from $\mathbb{N}_n$ to $\mathbb{N}_k$ is:

$$|S| = k^n.$$

- For $|A_{i_1}|$: The number of maps where a specific element y_{i_1} is not mapped to is:

  $$|A_{i_1}| = (k-1)^n.$$

  There are $\binom{k}{1} = k$ such terms.

- For $|A_{i_1} \cap A_{i_2}|$: The number of maps where two specific elements y_{i_1} and y_{i_2} are not mapped to is:

  $$|A_{i_1} \cap A_{i_2}| = (k-2)^n.$$

  There are $\binom{k}{2}$ such terms.

- In general, for the intersection of m sets $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$, the number of maps where m specific elements are not mapped to is:

  $$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = (k-m)^n.$$

  There are $\binom{k}{m}$ such terms.

Substituting all the computed values into the inclusion-exclusion formula, we get:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}| = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots + (-1)^k \binom{k}{k}(k-k)^n.$$

Simplifying, this becomes:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}| = \sum_{m=0}^{k} (-1)^m \binom{k}{m}(k-m)^n.$$

The number of surjections from $\mathbb{N}_n$ to $\mathbb{N}_k$ is:

$$\boxed{\sum_{m=0}^{k} (-1)^m \binom{k}{m}(k-m)^n}.$$

Answers to Exercise 20:

Counting Permutations Without Forbidden Sequences

We are tasked with determining the number of permutations of the 26 letters of the alphabet that do not contain any of the following sequences: "MATH", "EXA", "DISCR", or "NHSM". To solve this, we employ the inclusion-exclusion principle.

We aim to calculate:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |S| - \sum_{k=1}^{4} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq 4} |A_{i_1} \cap \cdots \cap A_{i_k}|,$$

where:

- $\overline{A_1}$: Set of permutations that do not contain "MATH",
- $\overline{A_2}$: Set of permutations that do not contain "EXA",
- $\overline{A_3}$: Set of permutations that do not contain "DISCR",
- $\overline{A_4}$: Set of permutations that do not contain "NHSM".

Therefore:

- A_1: Set of permutations containing "MATH",
- A_2: Set of permutations containing "EXA",
- A_3: Set of permutations containing "DISCR",
- A_4: Set of permutations containing "NHSM".

Computing Individual Terms and Intersections

The total number of permutations of 26 letters is:

$$|S| = 26!.$$

- $|A_1| = 23!$ (arranging 23 units: the "MATH" block + 22 remaining letters),
- $|A_2| = 24!$ (arranging 24 units: the "EXA" block + 23 remaining letters),
- $|A_3| = 22!$ (arranging 22 units: the "DISCR" block + 21 remaining letters),
- $|A_4| = 23!$ (arranging 23 units: the "NHSM" block + 22 remaining letters).

Pairwise Intersections

$$|A_1 \cap A_2| = 0, \quad |A_1 \cap A_3| = 19!, \quad |A_1 \cap A_4| = 0,$$
$$|A_2 \cap A_3| = 20!, \quad |A_2 \cap A_4| = 21!, \quad |A_3 \cap A_4| = 0.$$

All intersections of three or more sets are zero:

$$|A_i \cap A_j \cap A_k| = 0, \quad |A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$$

General Rule for Pairwise Intersections

- If the sequences do not share any letters, treat them as separate blocks and permute with the remaining letters.
- If the sequences share letters:

– If the shared letters can overlap (e.g., **EMA** and **MATH** form **EMATH**), treat them as a single combined block:

$$|A_i \cap A_j| = (n - k_1 - k_2 + 1)!,$$

where $n = 26$, and k_1, k_2 are the lengths of the sequences.

– If the shared letters cannot overlap (e.g., **MATH** and **NHSM** share H in conflicting positions), the intersection is:

$$|A_i \cap A_j| = 0.$$

Substituting the computed values into the inclusion-exclusion formula:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 26! - (23! + 24! + 22! + 23!) + (19! + 20! + 21!).$$

Answers to Exercise 21:

See Chapter 2 lectures.

Answers to Exercise 22:

We seek to find:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|,$$

which represents the number of permutations that do not contain any of the consecutive pairs **"ee"**, **"ss"**, or **"tt"**. Let:

- A_1 be the set of permutations containing **"ee"**.
- A_2 be the set of permutations containing **"ss"**.
- A_3 be the set of permutations containing **"tt"**.

The word **"discretemaths"** has the following letter frequencies:

d: 1, i: 1, s: 2, c: 1, r: 1, e: 2, t: 2, m: 1, a: 1, h: 1.

The total number of letters is 13. The total number of permutations of the word, without any restrictions, is:

$$|S| = \frac{13!}{2! \cdot 2! \cdot 2!}.$$

Using the inclusion-exclusion principle, we have:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

- For $|A_1|$: Treat **"ee"** as a single entity. The word now has 12 entities. The number of permutations is:

$$|A_1| = \frac{12!}{2! \cdot 2!}.$$

- For $|A_2|$: Treat **"ss"** as a single entity. The word now has 12 entities. The number of permutations is:

$$|A_2| = \frac{12!}{2! \cdot 2!}.$$

- For $|A_3|$: Treat **"tt"** as a single entity. The word now has 12 entities. The number of permutations is:

$$|A_3| = \frac{12!}{2! \cdot 2!}.$$

- For $|A_1 \cap A_2|$: Treat **"ee"** and **"ss"** as single entities. The word now has 11 entities. The number of permutations is:

$$|A_1 \cap A_2| = \frac{11!}{2!}.$$

- For $|A_1 \cap A_3|$: Treat **"ee"** and **"tt"** as single entities. The word now has 11 entities. The number of permutations is:
$$|A_1 \cap A_3| = \frac{11!}{2!}.$$
- For $|A_2 \cap A_3|$: Treat **"ss"** and **"tt"** as single entities. The word now has 11 entities. The number of permutations is:
$$|A_2 \cap A_3| = \frac{11!}{2!}.$$

Compute $|A_1 \cap A_2 \cap A_3|$

Treat **"ee"**, **"ss"**, and **"tt"** as single entities. The word now has 10 entities. The number of permutations is:

$$|A_1 \cap A_2 \cap A_3| = 10!.$$

Substituting all the computed values into the inclusion-exclusion formula, we get:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = \boxed{\frac{13!}{2!\cdot 2!\cdot 2!} - 3\cdot\frac{12!}{2!\cdot 2!} + 3\cdot\frac{11!}{2!} - 10!}.$$

Answers to Exercise 23:

We seek to find the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r,$$

with the constraints $0 \le x_i \le k$ for each $i = 1, 2, \ldots, n$.
We seek to find:
$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|,$$
where $\overline{A_i}$ denotes the subset of solutions satisfying $0 \le x_i \le k$.
Using the inclusion-exclusion principle (IEP), we have:

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= |S| - \sum_{m=1}^{n}(-1)^{m-1}\sum_{1\le i_1<\cdots<i_m\le n}|A_{i_1}\cap\cdots\cap A_{i_m}|,\\ &= |S| - \sum_{i_1=1}^{n}|A_{i_1}| + \sum_{1\le i_1<i_2\le n}|A_{i_1}\cap A_{i_2}| - \cdots + (-1)^n|A_1\cap A_2\cap\cdots\cap A_n|,\end{aligned}$$

where:

- S is the set of all nonnegative integer solutions to $x_1 + x_2 + \cdots + x_n = r$.
- A_i is the set of solutions where $x_i \ge k+1$.

The total number of nonnegative integer solutions without constraints is:

$$|S| = \left(\!\!\binom{n}{r}\!\!\right) = \binom{r+n-1}{r}.$$

For each i, the number of solutions where $x_i \ge k+1$ is:

$$|A_i| = \left(\!\!\binom{n}{r-(k+1)}\!\!\right) = \binom{r-(k+1)+n-1}{r-(k+1)}.$$

Since all the constraints are the same then

$$\sum_{1\le i_1\le n}|A_{i_1}| = \binom{n}{1}\left(\!\!\binom{n}{r-(k+1)}\!\!\right)$$

For each pair $i_1 < i_2$, the number of solutions where $x_{i_1} \geq k+1$ and $x_{i_2} \geq k+1$ is:

$$|A_{i_1} \cap A_{i_2}| = \left(\!\!\binom{n}{r-2(k+1)}\!\!\right) = \binom{r-2(k+1)+n-1}{r-2(k+1)}.$$

Since all the constraints are the same then

$$\sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| = \binom{n}{2} \left(\!\!\binom{n}{r-2(k+1)}\!\!\right)$$

Similarly, for intersections of m subsets $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$, the number of solutions is:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \left(\!\!\binom{n}{r-m(k+1)}\!\!\right) = \binom{r-m(k+1)+n-1}{r-m(k+1)}.$$

and

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \binom{n}{m} \left(\!\!\binom{n}{r-m(k+1)}\!\!\right)$$

Substituting into the IEP formula, we obtain:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(\!\!\binom{n}{r-m(k+1)}\!\!\right) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \binom{r-m(k+1)+n-1}{r-m(k+1)}.$$

The number of integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$ with $0 \leq x_i \leq k$ for each $i = 1, 2, \ldots, n$ is:

$$\boxed{\sum_{m=0}^{n} (-1)^m \binom{n}{m} \binom{r-m(k+1)+n-1}{r-m(k+1)}}.$$

Answers to Exercise 24:

1. We seek to count the number of r-element subsets of A that contain B as a subset. Since B is already a subset of these r-element subsets, we only need to choose the remaining $r-m$ elements from the $n-m$ elements of $A \setminus B$.

   The number of ways to choose $r-m$ elements from $n-m$ elements is:

$$\boxed{\binom{n-m}{r-m} = \binom{n-m}{n-r}}.$$

2. We will prove the identity using the inclusion-exclusion principle. Let A be a set with $|A| = n$, and let $B \subseteq A$ with $|B| = m$. We seek to count the number of r-element subsets of A that contain all elements of B.

   Let:

   - S be the set of all r-element subsets of A.
   - For each $i \in B$, let A_i be the set of r-element subsets of A that do not contain i.

   We seek to find:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}|,$$

   which represents the number of r-element subsets of A that contain all elements of B.

   The total number of r-element subsets of A is:

$$|S| = \binom{n}{r}.$$

Using the inclusion-exclusion principle, we have:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}| = |S| - \sum_{i_1=1}^{m} |A_{i_1}| + \sum_{1 \le i_1 < i_2 \le m} |A_{i_1} \cap A_{i_2}| - \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|.$$

- For $|A_{i_1}|$: The number of r-element subsets that do not contain a specific element $i_1 \in B$ is:

$$|A_{i_1}| = \binom{n-1}{r}.$$

  There are $\binom{m}{1}$ such terms.
- For $|A_{i_1} \cap A_{i_2}|$: The number of r-element subsets that do not contain two specific elements $i_1, i_2 \in B$ is:

$$|A_{i_1} \cap A_{i_2}| = \binom{n-2}{r}.$$

  There are $\binom{m}{2}$ such terms.
- In general, for the intersection of i sets $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_i}$, the number of r-element subsets that do not contain i specific elements of B is:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_i}| = \binom{n-i}{r}.$$

  There are $\binom{m}{i}$ such terms.

Substituting all the computed values into the inclusion-exclusion formula, we get:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}| = \binom{n}{r} - \binom{m}{1}\binom{n-1}{r} + \binom{m}{2}\binom{n-2}{r} - \cdots + (-1)^m \binom{m}{m}\binom{n-m}{r}.$$

This can be written compactly as:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_m}| = \sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{n-i}{r}.$$

The number of r-element subsets of A that contain all elements of B is:

$$\binom{n-m}{r-m},$$

since we are choosing the remaining $r-m$ elements from the $n-m$ elements of $A \setminus B$.

Thus, we have:

$$\boxed{\sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{n-i}{r} = \binom{n-m}{r-m}}.$$

Answers to Exercise 25:

The right-hand side (RHS) is the same as the solution for Exercise 23 when $k = 1$. For the left-hand side (LHS), we consider the equation:

$$x_1 + x_2 + \cdots + x_n = k,$$

with the constraint $0 \le x_i \le 1$. This means that each x_i can only be either 0 or 1. Therefore, we need to choose exactly k of the x_i's to be 1, and the remaining $n-k$ will be 0. The number of solutions is given by:

$$\boxed{\binom{n}{k}}.$$

Answers to Exercise 27:

We seek to find the number of ways W_n of seating n married couples at a straight table such that no woman is seated next to her husband. This is a classic problem that can be solved using the **inclusion-exclusion principle (IEP)**.

First, consider the total number of ways to arrange $2n$ people (the n couples) in a straight table. Since the table is straight, the number of possible arrangements is:

$$|S| = (2n)!.$$

Let A_i denote the set of arrangements where the i-th couple is seated next to each other. We want to count the number of arrangements where **none** of the couples are seated together, which is:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|.$$

Using the inclusion-exclusion principle, we can write:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

For each i, $|A_i|$ is the number of arrangements where the i-th couple is seated together. Treat the i-th couple as a single "block." This reduces the problem to arranging $2n-1$ "blocks" (the $n-1$ other individuals and the 1 block for the i-th couple). The number of arrangements is:

$$|A_i| = 2 \cdot (2n-1)!.$$

Here, the factor of 2 accounts for the two possible orders in which the couple can be seated (husband-wife or wife-husband).

Since there are n couples, the total contribution from single intersections is:

$$\sum_{i=1}^{n} |A_i| = \binom{n}{1} \cdot 2 \cdot (2n-1)!.$$

For each pair $i < j$, $|A_i \cap A_j|$ is the number of arrangements where both the i-th and j-th couples are seated together. Treat each of these couples as a single "block." This reduces the problem to arranging $2n-2$ "blocks" (the $n-2$ other individuals and the 2 blocks for the i-th and j-th couples). The number of arrangements is:

$$|A_i \cap A_j| = 2^2 \cdot (2n-2)!.$$

Here, the factor of 2^2 accounts for the two possible orders for each of the two couples.

Since there are $\binom{n}{2}$ pairs of couples, the total contribution from double intersections is:

$$\sum_{1 \le i < j \le n} |A_i \cap A_j| = \binom{n}{2} \cdot 2^2 \cdot (2n-2)!.$$

For the intersection of m subsets $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$, the number of arrangements where m specific couples are seated together is:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = 2^m \cdot (2n-m)!.$$

Here, the factor of 2^m accounts for the two possible orders for each of the m couples.

Since there are $\binom{n}{m}$ ways to choose m couples, the total contribution from m-fold intersections is:

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \binom{n}{m} \cdot 2^m \cdot (2n-m)!.$$

Substituting into the inclusion-exclusion formula, we obtain:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \cdot 2^m \cdot (2n-m)!.$$

The number of ways W_n of seating n married couples at a straight table such that no woman is seated next to her husband is:

$$\boxed{W_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \cdot 2^m \cdot (2n-m)!}.$$