Worksheet 2: Sets and Maps

Answers to Exercise 9:

1. f_1 not injective and not surjective.
2. f_2 not injective and not surjective.
3. f_3 bijective and $f_3^{-1}(y) = \ln\left(-1+\sqrt{1+y}\right)$
4. f_4 bijective and $f_4^{-1}(y) = \frac{y}{1-|y|}$
5. f_5 not injective and surjective $\forall z \in \mathbb{R}$, if we take $(x, y) = (z/2, z/2) \implies f(x, y) = z$.
6. f_6 bijective and $f^{-1}(s, t) = (\frac{s+t}{2}, \frac{s-t}{2})$.
7. $f_7(x, y) = (x + y, xy)$,
   - **Injective (One-to-One):** The map f_7 is injective if

     $$\forall(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2; f(a_1, b_1) = f(a_2, b_2) \implies (a_1, b_1) = (a_2, b_2)$$

     .

     $$\begin{aligned}
     f(a_1, b_1) = f(a_2, b_2) &\iff (a_1 + b_1, a_1b_1) = (a_2 + b_2, a_2b_2)\\
     &\iff \begin{cases} a_1 + b_1 &= a_2 + b_2 \quad (1)\\ a_1b_1 &= a_2b_2 \quad (2)\end{cases}\\
     &\iff (a_1 + b_1)^2 = (a_2 + b_2)^2\\
     &\iff a_1^2 + b_1^2 + 2a_1b_1 = a_2^2 + b_2^2 + 2a_2b_2\\
     &\iff a_1^2 + b_1^2 - 2a_2b_2 = a_2^2 + b_2^2 - 2a_1b_1\\
     &\overset{\text{from (2)}}{\iff} a_1^2 + b_1^2 - 2a_1b_1 = a_2^2 + b_2^2 - 2a_2b_2\\
     &\iff (a_1 - b_1)^2 = (a_2 - b_2)^2\\
     &\iff (a_1 - b_1) = (a_2 - b_2) \text{ or } (a_1 - b_1) = (b_2 - a_2) \quad (3)
     \end{aligned}$$

     So by adding (1) and both equations of (3), we get

     $$\begin{aligned}
     &2a_1 = 2a_2 \text{ or } 2a_1 = 2b_2\\
     \iff &a_1 = a_2 \text{ or } a_1 = b_2\\
     \overset{\text{replace in (1)}}{\Rightarrow} &a_1 = a_2 \text{ and } b_1 = b_2 \text{ or } a_1 = b_2 \text{ and } b_1 = a_2;
     \end{aligned}$$

     However, it's possible that $(a_1 = b_2 \text{ and } b_1 = a_2) \iff (a_1, b_1) \neq (a_2, b_2)$. So, the function is **not injective**.
   - **Surjective (Onto):** The map f_7 If $\forall(c, d) \in \mathbb{R}^2, \exists(a, b) \in \mathbb{R}^2; f(a, b) = (c, d)$.
     Consider (c, d) in the codomain and find (a, b) such that $f(a, b) = (c, d)$.

     $$\begin{aligned}
     \begin{cases} a + b &= c\\ ab &= d\end{cases} &\iff \begin{cases} a &= c - b\\ ab &= d\end{cases} \iff \begin{cases} a &= c - b\\ (c - b)b &= d\end{cases} \iff \begin{cases} a &= c - b\\ -b^2 + cb - d &= 0\end{cases}\\
     &\iff \begin{cases} a &= c - b\\ b &= \frac{-c\pm\sqrt{c^2-4d}}{-2}\end{cases}\\
     &\iff \begin{cases} a &= \frac{-3c\pm\sqrt{c^2-4d}}{-2}\\ b &= \frac{-c\pm\sqrt{c^2-4d}}{-2}\end{cases}
     \end{aligned}$$

     So to find (a, b), c and d should satisfy both the inequality $c^2 - 4d \geq 0$ for arbitrary c and d. This might not be possible (for example if $c = d = 1$), and hence the function is not surjective.
   - **Bijective:** A function is bijective if it is both injective and surjective. Since the given function is not injective or surjective, it is not bijective.

8. $f_8(n,p) = 2^n(2p+1)$,
   - Let's prove that f_8 is injective.
     Consider two pairs (n,p) and (m,q) in $\mathbb{N}^2$ such that $f_8(n,p) = f_8(m,q)$. Then we have:
     $$2^n(2p+1) = 2^m(2q+1).$$
     Assume, for example, that $n \geq m$. We can rewrite this as:
     $$2^{n-m}(2p+1) = 2q+1.$$
     If $n \neq m$, the left term is even, and the right term is odd, which is a contradiction. So, we must have $n = m$. Thus, we obtain:
     $$2p+1 = 2q+1,$$
     which implies $p = q$. In conclusion, we have shown that $(n,p) = (m,q)$, and therefore, f_8 is injective.
   - Next, let's prove that f_8 is surjective. Take any $l \in \mathbb{N}$. We can write l as a product of prime factors:
     $$l = 2^n p_2^{\alpha_2} \dots p_r^{\alpha_r},$$
     where p_i for $i \geq 2$ are odd prime numbers. Since the product of odd numbers is also odd, we can express $p_2^{\alpha_2} \dots p_r^{\alpha_r}$ as $2p+1$, where $p \in \mathbb{N}$. Therefore, we have $l = 2^n(2p+1) = f_8(n,p)$, which means that f is surjective.
   - Thus, f_8 is a bijective function, as it is both injective and surjective. To obtain f_8^-, we factorize any element $m \in \mathbb{N}$ into its prime factors. We then determine n as the exponent of 2 in the prime factorization, and p as the quotient of the product of the odd primes divided by 2.

Answers to Exercise 10:

1. **Injectivity (One-to-One):**
   Let's assume that x_1 and x_2 are two distinct real numbers. This implies:
   $$f(x_1) = \frac{x_1}{\sqrt{1+x_1^2}} \quad \text{and} \quad f(x_2) = \frac{x_2}{\sqrt{1+x_2^2}}$$
   It's clear that x_1 and x_2 should be of the same sign since $f(x)$ has the same sign as x. We can then proceed as follows:
   $$\begin{aligned} & \frac{x_1}{\sqrt{1+x_1^2}} = \frac{x_2}{\sqrt{1+x_2^2}} \\ \iff & x_1^2(1+x_2^2) = x_2^2(1+x_1^2) \\ \iff & x_1^2 = x_2^2 \\ \iff & x_1 = x_2 \quad \text{(because the sign of } x_1 \text{ is the same as } x_2) \end{aligned}$$
   This shows that for all $x_1, x_2 \in \mathbb{R}$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Therefore, $f(x)$ is injective.
   Surjectivity (Onto):
   To determine whether $f(x)$ is surjective, we need to check if, for all $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$. We can proceed as follows:

$$\begin{aligned}\frac{x}{\sqrt{1+x^2}} = y &\iff x = y\sqrt{1+x^2}\\ &\iff x^2(1-y^2) = y^2\\ &\iff |x| = \frac{|y|}{\sqrt{1-y^2}}\\ &\iff x = \frac{y}{\sqrt{1-y^2}}\end{aligned}$$

So, the function f^- exists only if $-1 < y < 1$. Therefore, there exist values of y for which there are no corresponding values of x that satisfy $f(x) = y$. This means that $f(x)$ is not surjective.

2. • Let's determine $f^{-1}([1, +\infty[)$:

$$f^{-1}([1, +\infty[) = \{x \in \mathbb{R} \mid f(x) \in A\}$$

We have:

$$f(x) \in A \iff f(x) \geq 1 \iff \frac{x}{\sqrt{1+x^2}} \iff x^2 \geq 1 + x^2.$$

However, this condition is impossible for all values of x. Therefore,

$$f^{-1}([1, +\infty[) = \emptyset.$$

• Let's determine $f([1, 2])$:

$$f([1, 2]) = \{f(x) \in \mathbb{R} \mid x \in [1, 2]\}$$

From the previous result, we know that f is injective, and it's strictly monotonic on $\mathbb{R}$. Since $1 < 2$ and $f(1) = \frac{1}{\sqrt{2}} < f(2) = \frac{2}{\sqrt{5}}$, we can conclude that f is strictly decreasing. Therefore,

$$f([1, 2]) = [f(1), f(2)] = \left[\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}}\right].$$

• Let's determine $f([-2, 2])$:
For the same reasons as the previous question, we have:

$$f([-2, 2]) = [f(-2), f(2)] = \left[\frac{-2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right].$$

3. **Bijectivity of $f|^{]-1,1[}$:**
From the first question, we've established injectivity. The corestriction of f satisfies that for all $y \in]-1, 1[$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Therefore, f is bijective, and its inverse is $f^{-1}(y) = \frac{y}{\sqrt{1-y^2}}$.

We can easily verify that $f(f^{-1}(y)) = y$:

$$f(f^{-1}(y)) = f\left(\frac{y}{\sqrt{1-y^2}}\right) = \frac{\frac{y}{\sqrt{1-y^2}}}{\sqrt{1+\frac{y^2}{1-y^2}}} = \frac{y}{\sqrt{1-y^2+y^2}} = y.$$

4. Let g be the restriction of $f|^{]-1,1[}$ to $]-1, 1[$.
For $n = 2$:

$$g^{(2)}(x) = g \circ g(x) = g(g(x)) = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{\sqrt{1+x^2}}}} = \frac{x}{\sqrt{1+2x^2}}.$$

For $n=3$:

$$g^{(3)}(x)=g\circ g\circ g(x)=\frac{x}{\sqrt{1+3x^2}}.$$

Let's suppose by induction on n that for all $n\geq 1$:

$$g^{(n)}(x)=\underbrace{g\circ g\circ\cdots\circ g}_{n\text{ times}}(x)=\frac{x}{\sqrt{1+nx^2}}.$$

Let's show that $g^{(n+1)}(x)=\frac{x}{\sqrt{1+(n+1)x^2}}$,

$$\begin{aligned}g^{(n+1)}(x)=g(g^{(n)}(x))&=g\left(\frac{x}{\sqrt{1+nx^2}}\right)\\&=\frac{\frac{x}{\sqrt{1+nx^2}}}{\sqrt{1+\frac{x^2}{1+nx^2}}}\\&=\frac{x}{\sqrt{1+nx^2+x^2}}\\&=\frac{x}{\sqrt{1+(n+1)x^2}}.\end{aligned}$$

This proves the result.

Answers to Exercise 11:

See the page 43 of the book.

Answers to Exercise 12:

1. • $A=\{\frac{1}{3},4\}$ $f^{-1}(A)$:
   To find the preimage of set A under the function f, you need to find all values in the domain such that $f(n)$ is in set A.

$$f^{-1}(A)=\{n\in\mathbb{N}\backslash\{1\}|f(n)\in A\}\iff f^{-1}(A)=\{n\in\mathbb{N}\backslash\{1\}|f(n)=\frac{1}{3}\text{ and }f(n)=4\}$$

   Solving the following equations will gives the values of n for which $f(n)$ belongs to set A: $\frac{n^2-n+2}{n-1}=\frac{1}{3}$ and $\frac{n^2-n+2}{n-1}=4$

   – For $f(n)=\frac{1}{3}$

$$\frac{n^2-n+2}{n-1}=\frac{1}{3}\iff 3n^2-4n+7=0\iff\text{no solutions in }\mathbb{N}\backslash\{1\}.$$

   – For $f(n)=4$

$$\frac{n^2-n+2}{n-1}=4\iff n^2-5n+6=0\iff n=2\text{ and }n=3.$$

   Therefore, $f^{-1}(A)=\{2,3\}$.

   • $B=]-\infty,-1]$

$$\begin{aligned}f^{-1}(B)&=f^{-1}(]-\infty,-1])\\&=\{n\in\mathbb{N}\setminus\{1\}\mid f(n)\in B\}\\&=\{n\in\mathbb{N}\setminus\{1\}\mid\frac{n^2-n+2}{n-1}\leq-1\}\\&=\{n\in\mathbb{N}\setminus\{1\}\mid n^2\leq-1\}\\&=\emptyset\quad(\text{there's no positive }n^2\text{ less than }-1)\end{aligned}$$

   • $C=\{n\in\mathbb{N}\setminus\{1\}\mid|n^3-n+1|\leq 1\}$

We know that $\forall n \geq 2, n(n^2-1) \geq 0$, therefore $n^3 - n + 1 \geq 1$.
This means $C = \emptyset$. Then, $f(C) = f(\emptyset) = \emptyset$.

2. Let $H = \{x \in \mathbb{Q} | \frac{1}{x - E(x)} \in \mathbb{N}\}$,

$$f^{-1}(H) = \{n \in \mathbb{N}\backslash\{1\} | f(n) \in H\}.$$

So we should find n such that $f(n) \in H$,

$$\begin{aligned} f(n) \in H &\implies \frac{1}{f(n) - E(f(n))} \in \mathbb{N} \\ &\iff \frac{1}{\frac{n^2-n+2}{n-1} - E(\frac{n^2-n+2}{n-1})} \in \mathbb{N} \\ &\iff \frac{1}{n + \frac{2}{n-1} - E(n + \frac{2}{n-1})} \in \mathbb{N} \\ &\iff \frac{1}{n + \frac{2}{n-1} - n - E(\frac{2}{n-1})} \in \mathbb{N} \\ &\iff \frac{n-1}{2 - (n-1)E(\frac{2}{n-1})} \in \mathbb{N} \end{aligned}$$

It's clear if $n > 3$ then $E(\frac{2}{n-1}) = 0$, therefore for $n > 3$:

$$\frac{n-1}{2 - (n-1)E(\frac{2}{n-1})} \in \mathbb{N} \iff \frac{n-1}{2} \in \mathbb{N} \iff 2 \text{ divides } n-1 \iff n = 2k+1.$$

Now if $2 \leq n \leq 3$:

- For $n = 2$ we have $2 - (2-1)E(\frac{2}{2-1}) = 2 - 2 = 0$ so $\frac{n-1}{2-(n-1)E(\frac{2}{n-1})}$ is not defined.
- For $n = 3$ we have $f(3) = 4$ so $f(3) - E(f(3)) = 4 - 4 = 0$ then $\frac{1}{f(n)-E(f(n))}$ is not defined also;

So we deduce that $f^{-1}(H) = \{2k+1 | k \in \mathbb{N}, k \geq 2\}$.

Answers to Exercise 14:

Let X, Y and Z be three sets and let $f : X \to Y$ and $g : Y \to Z$ be two maps.

- If $g \circ f$ is injective then f is injective.
  Let $x, x' \in X$ such that $f(x) = f(x')$.

$$\begin{aligned} f(x) = f(x') &\overset{g \text{ is a map}}{\implies} g(f(x)) = g(f(x')) \\ &\iff g \circ f(x) = g \circ f(x') \\ &\overset{g\circ f \text{ is injective}}{\iff} x = x'. \end{aligned}$$

  So $\forall x, x' \in X, f(x) = f(x') \implies x = x'$. Therefore, if $g \circ f$ is injective then f is injective.

- If $g \circ f$ is injective and f is surjective then g injective. Let $y, y' \in F$ such that $g(y) = g(y')$; to proof that g is injective we need to show that $y = y'$. Hence we know that f is surjective so

$$\begin{aligned} g(y) = g(y') &\overset{f \text{ surj}}{\implies} \exists x, x' \in X; f(x) = y, f(x') = y' \;\&\; g(f(x)) = g(f(x')) \\ &\overset{\text{def}}{\iff} \exists x, x' \in X; g \circ f(x) = g \circ f(x') \\ &\overset{g\circ f \text{inj}}{\implies} x = x' \\ &\overset{\text{f is map}}{\implies} f(x) = f(x') \\ &\iff y = y'. \end{aligned}$$

  Therefore g is injective.

- If $g \circ f$ is surjective then g is surjective. Let $z \in Z$, hence $g \circ f$ is surjective, so

$$\begin{aligned} \exists x \in X; g \circ f(x) = z &\iff \exists x \in X; g(f(x)) = z; \\ &\overset{\text{put } f(x)=y}{\iff} \forall z \in Z \exists y \in Y, \exists x \in X; g(y) = z \& \ f(x) = y; \end{aligned}$$

Therefore g is surjective.

- If $g \circ f$ is surjective and g is injective then f is surjective. Let $y \in Y$,

$$\begin{aligned} y \in Y &\overset{g \text{ is map}}{\Longrightarrow} g(y) \in Z \\ &\overset{g \circ f \text{ is surj}}{\Longrightarrow} \exists x \in X, g \circ f(x) = g(y) \\ &\overset{\text{def}}{\iff} \exists x \in X, g(f(x)) = g(y) \\ &\overset{g \text{ inj}}{\Longrightarrow} \exists x \in X, f(x) = y. \end{aligned}$$

Then f is surjective.

- For any subset A of $X, A \subset f^{-1}(f(A))$.

$$\begin{aligned} \text{Let } x \in A &\Longrightarrow f(x) \in f(A) && \text{(from definition of image of a set)} \\ &\Longrightarrow x \in f^{-1}(f(A)) && \text{(from definition of preimage of a set)} \end{aligned}$$

That's means $A \subset f^{-1}(f(A))$.

- f is injective if and only if for any subset A of X, $f^{-1}(f(A)) \subset A$.

$\Longrightarrow$): f injective $\Longrightarrow f^{-1}(f(A)) \subset A$

$$\begin{aligned} \text{Let } x \in f^{-1}(f(A)) &\Longrightarrow \exists x' \in A; f(x) = f(x') \& \ f(x') \in f(A) && \text{(from definition of preimage of a set)} \\ &\overset{f \text{ inj}}{\Longrightarrow} x = x' \\ &\Longrightarrow x \in A. \end{aligned}$$

$\Longleftarrow$): $f^{-1}(f(A)) \subset A \Longrightarrow f$ injective. Let $x, x' \in X$ such that $f(x) = f(x')$, and let $A = \{x\} \subset X$. So

$$f(A) = f(\{x\}) = \{f(x)\} \overset{f(x)=f(x')}{=} \{f(x')\} \Longrightarrow x' \in f^{-1}(f(A)) \overset{hyp}{\Longrightarrow} x' \in A \Longrightarrow x = x'.$$

Then f is injective.

- For any subset B of Y , $f(f^{-1}(B)) \subset B$.

$$\begin{aligned} \text{Let } y \in f(f^{-1}(B)) &\Longrightarrow \exists x \in f^{-1}(B) \& f(x) = y && \text{(from definition of image of a set)} \\ &\Longrightarrow f(x) \in B \& \ f(x) = y && \text{(from definition of preimage of a set)} \\ &\iff y \in B. \end{aligned}$$

Then $f(f^{-1}(B)) \subset B$.

- f is surjective if and only if for any subset B of $Y, B \subset f(f^{-1}(B))$.

$\Longrightarrow$): f surjective $\Longrightarrow B \subset f(f^{-1}(B))$

$$\begin{aligned} \text{Let } y \in B &\overset{\text{f surj}}{\Longrightarrow} \exists x \in X; y = f(x) \in B \\ &\Longrightarrow x \in f^{-1}(B) \\ &\Longrightarrow f(x) \in f(f^{-1}(B)) \iff y \in f(f^{-1}(B)). \end{aligned}$$

$\Longleftarrow$): $B \subset f(f^{-1}(B)) \Longrightarrow f$ surjective. Let $y \in Y$, and let $B = \{y\} \subset Y$. From the hypothesis we have:

$$B \subset f(f^{-1}(B)) \Longrightarrow y \in f(f^{-1}(B)) \Longrightarrow \forall y \in Y, \exists x \in f^{-1}(B) \subset X; f(x) = y.$$

Then f is surjective.