Chapter 4: Integer Partitions

1. Introduction to Integer Partitions

Definition 1.1 (Integer Partition).
A **partition** of a non-negative integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$$

and

$$\sum_{i=1}^{k} \lambda_i := \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

The integers λ_i are called the **parts** of the partition. We write $\lambda \vdash n$ to denote that λ is a partition of n. The number of parts is k.

The number of distinct partitions of n is denoted by $p(n)$. By convention, $p(0) = 1$ (representing the empty partition, where $k = 0$).

Example.
For $n = 4$, $p(4) = 5$. The partitions are: (4), (3,1), (2,2), (2,1,1), (1,1,1,1).

Proposition 1.2.
The number of partitions of n, $p(n)$, is equal to the number of non-negative integer solutions $(m_1, m_2, \ldots, m_n)$ to the equation:

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n, \quad \text{where } m_i \geq 0 \text{ for all } i.$$

Here, m_i represents the number of times the integer i appears as a part in the partition.

2. Visualizing Partitions: Ferrers Diagrams

Definition 2.1 (Ferrers Diagram).
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ can be visualized using a **Ferrers diagram** (or Young diagram). It consists of k rows of boxes (or dots), where the i-th row has λ_i boxes. The rows are left-aligned.

Example.
The partition $(5, 3, 3, 1)$ of 12 can be represented.

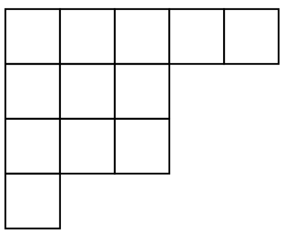

Figure 5: A representation for the partition (5,3,3,1).

Definition 2.2 (Conjugate Partition).
The **conjugate** of a partition λ, denoted λ', is the partition whose Ferrers diagram is obtained by transposing the Ferrers diagram of λ (i.e., swapping rows and columns).

Example.
The conjugate of the partition $\lambda = (5, 3, 3, 1)$ of 12 is $\lambda' = (4, 3, 3, 1, 1)$, and can be represented.

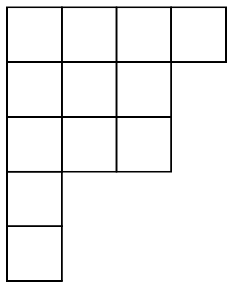

Figure 6: A representation for the partition (4,3,3,1,1).

3. Generating Functions for Partitions

The **generating function** for $p(n)$ is $P(x) = \sum_{n=0}^{\infty} p(n)x^n$, where the coefficient $p(n)$ represent the number of partitions of n.

Theorem 3.1 (Euler's Generating Function for $p(n)$)**.**

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots) \cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

4. Restricted Partitions and Euler's Partition Theorem

4.1 Partitions into Exactly k Parts

Definition 4.1.
Let $p(n, k)$ denote the number of partitions of n into exactly k parts.

Example.
For $n = 5, k = 2$: partitions are $(4, 1), (3, 2)$. So $p(5, 2) = 2$.

Proposition 4.2 (Recurrence for $p(n, k)$)**.**
For $n, k \geq 1$: $p(n, k) = p(n-k, k-1) + p(n-k, k) = p(n-1, k-1) + p(n-k, k) = \sum_{i=1}^{min(k,n-k)} p(n-k, i)$.

The table below lists $p(n, k)$:

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0
5	0	1	2	2	1	1	0	0	0
6	0	1	3	3	2	1	1	0	0
7	0	1	3	4	3	2	1	1	0
8	0	1	4	5	5	3	2	1	1

Theorem 4.3.
The number of partitions of n into k parts is equal to the number of partitions of n where the largest part is k.

4.1.1 Partitions into at most k Parts

Definition 4.4.
Let $p_{\leq k}(n)$ denote the number "partitions of n with at most k parts."

Example.
Let $n = 5$ and $k = 2$. We are looking for partitions of 5 with at most 2 parts.

- Partitions with exactly 1 part ($j = 1$): (5). There is $p(5, 1) = 1$ such partition.
- Partitions with exactly 2 parts ($j = 2$): (4,1), (3,2). There are $p(5, 2) = 2$ such partitions.

So, $p_{\leq 2}(5) = p(5, 1) + p(5, 2) = 1 + 2 = 3$.

Proposition 4.5.
The function $p_{\leq k}(n)$ (counting partitions of n into at most k parts) has the following properties:

1. For $n \geq 0$:
$$p_{\leq k}(n) = \sum_{j=0}^{k} p(n, j)$$
2. For $k \geq 1$ and $n \geq 0$:
$$p(n, k) = p_{\leq k}(n) - p_{\leq (k-1)}(n)$$
3. If $k \geq n$:
$$p_{\leq k}(n) = p(n)$$

4.1.2 Generating function of $p_{\leq k}(n)$ and $p(n, k)$

Theorem 4.6 (Generating Functions by Number of Parts)**.**
Let $p_{\leq k}(n)$ be the number of partitions of n into at most k positive parts, and let $p(n, k)$ be the number of partitions of n into exactly k positive parts. Their respective ordinary generating functions are:

1. For partitions into at most k parts:
$$P_{\leq k}(x) = \sum_{n=0}^{\infty} p_{\leq k}(n)x^n = \prod_{i=1}^{k} \frac{1}{1 - x^i}$$
2. For partitions into exactly k parts:
$$P_k(x) = \sum_{n=k}^{\infty} p(n, k)x^n = \prod_{i=1}^{k} \frac{x^k}{1 - x^i}$$
(The sum for $P_k(x)$ starts at $n = k$ as n must be at least k to have k positive parts.)

4.2 General Approach to Restricted Partitions

Generating functions can be adapted to count partitions with various restrictions. If a part j can appear s_1 or s_2 or ... s_m times, its factor in the generating function is $(x^{js_1} + x^{js_2} + \cdots + x^{js_m})$. The overall generating function is the product of these factors.

4.2.1 Partitions into Distinct Parts and Odd Parts

Example (Generating Function for Partitions into Distinct Parts)**.**
Let $p_d(n)$ be the number of partitions of n where all parts are distinct. Each part k can appear 0 or 1 time. Factor for part k: $(1 + x^k)$. Generating function: $P_d(x) = \prod_{k=1}^{\infty}(1 + x^k)$.

Example (Generating Function for Partitions into Odd Parts)**.**
Let $p_o(n)$ be the number of partitions of n where all parts are odd. Only odd parts $j = 2k - 1$ are allowed, appearing any number of times. Factor for odd part j: $\frac{1}{1-x^j}$. Generating function: $P_o(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$.

Theorem 4.7 (Euler's Partition Theorem, 1748).
The number of partitions of n into distinct parts equals the number of partitions of n into odd parts: $p_d(n) = p_o(n)$ for all $n \geq 0$.

4.2.2 Further Examples of Restricted Partitions

Example (Partitions where the part '2' appears at most once).
Let $q(n)$ be the number of such partitions. Generating function: $Q(x) = (1 + x^2) \prod_{k \neq 2, k \geq 1} \frac{1}{1-x^k} = (1 - x^4)P(x)$. So, $q(n) = p(n) - p(n-4)$ (with $p(j) = 0$ if $j < 0$).

Example (Partitions where each part appears at most twice).
Let $r(n)$ be the number of such partitions. Factor for part k: $(1 + x^k + x^{2k})$. Generating function: $R(x) = \prod_{k=1}^{\infty}(1 + x^k + x^{2k}) = \prod_{k=1}^{\infty} \frac{1-x^{3k}}{1-x^k} = \frac{P(x)}{P(x^3)}$.

5. Exercises

Exercise.
List partitions of $n = 6$. Verify $p_d(6) = p_o(6)$. Calculate $p(6, 3)$ using its recurrence.

Exercise.
Draw the Ferrers diagram for $(5, 2, 2)$ and find its conjugate.

Exercise.
Write the generating function for partitions where all parts are ≥ 2 and distinct.

Appendix A: Ordinary and Exponential Generating Functions

Definition .1 (Generating Function).
The **Generating Function** for an infinite sequence $\{a_n\}_{n=0}^{\infty} = (a_0, a_1, a_2, \dots)$ is the formal power series:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

In this series, the coefficient a_n is the n-th term of the sequence, and the exponent n in x^n indicates its position. This series is often represented by a *compact expression* , and *expanding* this expression generate a power series having the terms of the sequence as coefficients.

Example.
Consider the sequence $(1, 1, 1, 1, \dots)$. Its generating function can be given by the compact expression $\frac{1}{1-x}$. When we expands this expression as a power series, we get:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

The coefficients of this expansion are $1, 1, 1, 1, \dots$, which are exactly the terms of our original sequence.

Definition .2 (Ordinary and Exponential Generating Function).
Given a sequence $(a_n)_{n\geq 0}$,

- The **Ordinary generating function (OGF)** is the formal power series:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- The **Exponential generating function (EGF)** is the formal power series:

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Definition .3 (Mixed Generating Function).
Given a double sequence $(a_{n,k})_{n,k\geq 0}$, its **mixed ordinary-exponential generating function** is:

$$A(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}\, x^n \,\frac{y^k}{k!}.$$

Theorem .4 (Generating Function for Constant and Linear Sequences).
Let $a_n = 1$ for all $n \geq 0$. Then:

$$\text{OGF: } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\text{EGF: } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Let $a_n = n$. Then:

$$\text{OGF: } \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$\text{EGF: } \sum_{n=0}^{\infty} \frac{n x^n}{n!} = x e^x$$

Theorem .5 (Generating Function of Fibonacci Numbers).
Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The generating function of the Fibonacci sequence is:

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$