Chapter 3: Other Important Combinatorial Sequences

1. Eulerian Numbers

Definition 1.1.
In a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ of the set $[n] = \{1, 2, \ldots, n\}$:

- An **ascent** is a position i (where $1 \le i \le n-1$) such that $\pi_i < \pi_{i+1}$.
- A **descent** is a position i (where $1 \le i \le n-1$) such that $\pi_i > \pi_{i+1}$.

Example.
Consider the permutation $\pi = 3142$:

- **Ascents**: Position 2 (since $1 < 4$).
- **Descents**: Positions 1 (since $3 > 1$) and 3 (since $4 > 2$).

Consider the permutation $\pi = 1234$:

- **Ascents**: Positions 1, 2, and 3 (all adjacent pairs are increasing).
- **Descents**: None.

Consider the permutation $\pi = 4321$:

- **Ascents**: None.
- **Descents**: Positions 1, 2, and 3 (all adjacent pairs are decreasing).

Theorem 1.2 (Complementarity of Ascents and Descents)**.**
For any permutation $\pi = \pi_1\pi_2\ldots\pi_n$ of $[n]$:

$$\text{Number of ascents} + \text{Number of descents} = n - 1.$$

Definition 1.3.
The **Eulerian number** $\left\langle {n \atop k} \right\rangle$ is the number of permutations of $[n]$ with exactly k ascents.

Example.
For $n = 3$:

- $\left\langle {3 \atop 0} \right\rangle = 1$ (permutation 321),
- $\left\langle {3 \atop 1} \right\rangle = 4$ (permutations $213, 132, 312, 231$),
- $\left\langle {3 \atop 2} \right\rangle = 1$ (permutation 123).

Theorem 1.4 (Recurrence for Eulerian Numbers)**.**
The Eulerian numbers satisfy the recurrence:

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle,$$

with boundary conditions $\left\langle {0 \atop 0} \right\rangle = 1$ and $\left\langle {n \atop k} \right\rangle = 0$ if $k < 0$ or $k \ge n$.

1.1 Special Values of Eulerian Numbers

For all $n \ge 1$:

- $\left\langle {n \atop 0} \right\rangle = 1$ (only the decreasing permutation)
- $\left\langle {n \atop n-1} \right\rangle = 1$ (only the increasing permutation)
- $\left\langle {n \atop 1} \right\rangle = 2^n - n - 1$

- $\left\langle {n \atop k} \right\rangle = 0 \quad \text{if } (k >= n)$

Theorem 1.5 (Symmetry).
$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$

Theorem 1.6 (Sum Identity).
For any integer $n \geq 1$,

$$\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n!$$

1.2 Triangle of Eulerian Numbers

Using the recurrence $\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$, the triangle is:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	0								
2	1	1	0							
3	1	4	1	0						
4	1	11	11	1	0					
5	1	26	66	26	1	0				
6	1	57	302	302	57	1	0			
7	1	120	1191	2416	1191	120	1	0		
8	1	247	4293	15619	15619	4293	247	1	0	
9	1	502	14608	88234	156190	88234	14608	502	1	0

1.3 Generating Function

The Eulerian numbers appear as coefficients in certain generating functions. The **Eulerian polynomial** $A_n(x)$ is defined as:

$$A_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$$

(Note: $\left\langle {n \atop k} \right\rangle = 0$ for $k > n-1$ when $n \geq 1$, so the sum effectively goes up to $k = n-1$. For $n = 0$, $A_0(x) = \left\langle {0 \atop 0} \right\rangle x^0 = 1$).

Proposition 1.7 (Recurrence Relation for Eulerian Polynomials).
For all integers $n \geq 1$, the Eulerian polynomials satisfy:

$$A_n(x) = (1 + (n-1)x)A_{n-1}(x) + (x - x^2)A'_{n-1}(x), \quad \text{for } n \geq 1,$$

with $A_0(x) = 1$.

Theorem 1.8 (Generating Function for Eulerian Polynomials).
The Eulerian polynomials $A_n(x)$ are related to the generating function for powers as follows:

$$\sum_{j=1}^{\infty} j^n x^j = \frac{xA_n(x)}{(1-x)^{n+1}}, \quad \text{for } n \geq 0. \quad (*)$$

Alternatively, a mixed generating function for Eulerian numbers is:

$$\mathcal{A}(z,x) = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k \frac{z^n}{n!} = \frac{1-x}{e^{z(x-1)} - x}. \quad (**)$$

1.4 Worpitzky's Identity

Worpitzky's identity provides a way to express monomials x^n as a sum involving Eulerian numbers and binomial coefficients.

Theorem 1.9 (Worpitzky's Identity).
For all integers $n \geq 0$ and variables x:

$$x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$$

with $\left\langle {n \atop n} \right\rangle = 0$ for $n \geq 1$.

Example (Worpitzky's Identity for n=2).
We have $A(2,0) = 1$, $A(2,1) = 1$, and $A(2,2) = 0$. According to Worpitzky's identity:

$$\begin{aligned} x^2 &= A(2,0)\binom{x+0}{2} + A(2,1)\binom{x+1}{2} + A(2,2)\binom{x+2}{2} \\ &= \binom{x}{2} + \binom{x+1}{2} + 0 \cdot \binom{x+2}{2} \\ &= \frac{x(x-1)}{2} + \frac{(x+1)x}{2} \\ &= \frac{x^2 - x + x^2 + x}{2} = \frac{2x^2}{2} = x^2. \end{aligned}$$

This demonstrates the identity for $n = 2$.

2. Harmonic Numbers and Related Sums

Definition 2.1 (Harmonic Numbers).
The n-th harmonic number is defined recursively by:

$$H_0 = 0, \quad H_n = H_{n-1} + \frac{1}{n} \quad \text{for } n \geq 1.$$

Theorem 2.2 (Relation to Stirling Numbers of the First Kind).
For $n \geq 1$, harmonic numbers satisfy:

$$H_n = \frac{1}{n!} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k,$$

where $\left[{n \atop k} \right]$ is the number of permutations of n elements with k cycles (unsigned Stirling numbers of the first kind).

Theorem 2.3 (Recurrence via Stirling Numbers).
The harmonic numbers also satisfy:

$$H_n = H_{n-1} + \frac{1}{n}.$$

3. Harmonic, Bernoulli, Euler, and Genocchi Numbers

This section introduces several other classical number sequences that appear frequently in combinatorics and analysis.

3.1 Harmonic Numbers

Definition 3.1.
The n-th **Harmonic Number**, denoted H_n, is defined as the sum of the reciprocals of the first n positive integers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad \text{for } n \geq 1.$$

By convention, $H_0 = 0$.

Example.
- $H_1 = 1$
- $H_2 = 1 + \frac{1}{2} = \frac{3}{2}$
- $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$
- $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

Theorem 3.2 (Recurrence for Harmonic Numbers)**.**
The Harmonic numbers satisfy the recurrence relation:

$$H_n = H_{n-1} + \frac{1}{n} \quad \text{for } n \geq 1,$$

with initial condition $H_0 = 0$.

Theorem 3.3 (Generating Function for Harmonic Numbers)**.**
The ordinary generating function for H_n is:

$$\sum_{n=1}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x}.$$

Theorem 3.4 (Relation with Stirling Numbers of the First Kind)**.**
For $n \geq 1$, the Harmonic numbers satisfy:

$$H_n = \frac{\left[{n+1 \atop 2}\right]}{n!},$$

where $\left[{n \atop k}\right]$ denotes the unsigned Stirling numbers of the first kind.

3.2 Bernoulli Numbers

Definition 3.5.
The **Bernoulli numbers**, denoted B_n, are defined recursively by:

$$B_n = \frac{-1}{(n+1)} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k = 0 \ \text{ for } \ n \geq 1.$$

with $B_0 = 1$.

Definition 3.6.
The Bernoulli numbers can alternatively be defined by the exponential generating function:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

The first few Bernoulli numbers are:

- $B_0 = 1$
- $B_1 = -\frac{1}{2}$ (Note: some older conventions use $B_1 = +\frac{1}{2}$)
- $B_2 = \frac{1}{6}$

- $B_3 = 0$
- $B_4 = -\frac{1}{30}$
- $B_5 = 0$
- $B_6 = \frac{1}{42}$

Proposition 3.7.
For all integers $k \geq 1$, the Bernoulli numbers satisfy $B_{2k+1} = 0$.

Exercise.
Prove the proposition.

3.3 Euler Numbers

Definition 3.8 (Recurrence Relation).
The Euler numbers E_n are defined by the following recurrence:

$$E_n = \begin{cases} -\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k} E_{2k} & \text{if } n \text{ is even,} \\ \\ 0 & \text{else.} \end{cases}$$

with $E_0 = 1$

Definition 3.9 (Generating Function).
The exponential generating function for Euler numbers is:

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

Key Properties of Euler Numbers

- **Non-Zero Values:** $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$, $E_{10} = -50521$, ...
- **Zero Values:** $E_{2k+1} = 0$ for all $k \geq 0$
- **Integrality:** All non-zero E_{2k} are integers
- **Alternating Signs:** $E_{4k} > 0$ and $E_{4k+2} < 0$ for $k \geq 0$

Proposition 3.10 (Combinatorial Interpretation).
The absolute value $|E_{2k}|$ (for $k \geq 0$) counts the number of **alternating permutations** of $\{1, 2, \ldots, 2k\}$ starting with an ascent. These permutations satisfy:

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots < \sigma_{2k-1} > \sigma_{2k}.$$

Special Case: For $k = 0$, $|E_0| = 1$ counts the empty permutation.

3.4 Genocchi Numbers

Definition 3.11 (Recurrence Relation).
The Genocchi numbers G_n are defined by the following recurrence:

$$G_n = 2(1 - 2^n) B_n$$

Definition 3.12 (Generating Function).
The exponential generating function for Genocchi numbers is:

$$\sum_{n=1}^{\infty} G_n \frac{x^n}{n!} = \frac{2x}{e^x + 1}.$$

Key Properties of Genocchi Numbers

- **Non-Zero Values:** $G_1 = 1$, $G_2 = -1$, $G_4 = 1$, $G_6 = -3$, $G_8 = 17$, $G_{10} = -155$, ...
- **Zero Values:** $G_{2k+1} = 0$ for $k \geq 1$
- **Integrality:** All non-zero G_{2k} are integers

Proposition 3.13 (Combinatorial Interpretation).
The absolute value $|G_{2k}|$ (for $k \geq 1$) counts the number of permutations of $[2n-1]$ with descents after the even numbers and ascents after the odd numbers.

Example.
For $k = 2$ ($2k-1 = 3$), the permutations of $\{1, 2, 3\}$ satisfying the conditions are:

$$\{2, 1, 3\} : \quad 2 > 1 < 3.$$

confirming $|G_2| = 1$. For $k = 3$ ($2k-1 = 5$), the permutations of $\{1, 2, 3, 5\}$ satisfying the conditions are:

$$\{\{2, 1, 4, 3, 5\}, \{3, 4, 2, 1, 5\}, \{4, 2, 1, 3, 5\}\}$$

confirming $|G_3| = 3$.

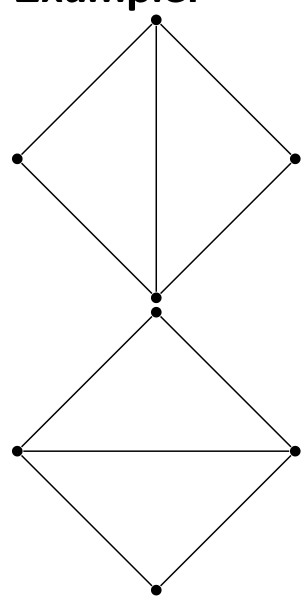
4. Catalan and Delannoy Numbers

4.1 Catalan Numbers

Definition 4.1 (Triangulation).
For a polygon with $n+2$ vertices, a *triangulation* is a set of $n-1$ non-intersecting diagonals that subdivides the polygon into n triangles.

Definition 4.2 (Catalan numbers).
The n-th *Catalan number* C_n is the number of triangulations of a convex polygon with $n+2$ vertices. We set $C_0 = 1$.

Example.



Theorem 4.3 (Catalan convolution via triangulations).
For every integer $n \geq 0$,

$$C_{n+1} = \sum_{k=0}^{n} C_k \, C_{n-k}.$$

4.1.1 Other Combinatorial Interpretations

Catalan numbers have a wide variety of interpretations in combinatorics. Here are a few common ones:

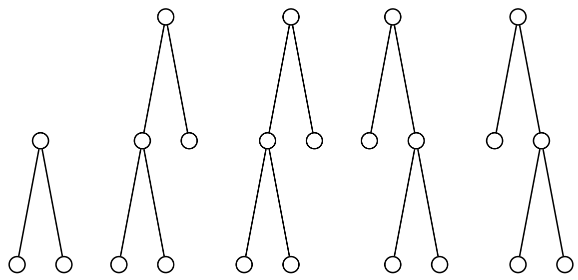
Proposition 4.4 (Parenthesizations)**.**
The number of distinct ways to fully parenthesize a product of $n+1$ terms is the n-th Catalan number C_n.

Example $(n = 2)$**:** For abc:

- $(a(bc))$
- $((ab)c)$

Proposition 4.5.
The number of (rooted) binary trees with n vertices—each vertex having a left and a right subtree that may be empty—is the n-th Catalan number C_n.

Example.
For $n = 3$ we have $C_3 = 5$ distinct binary trees. The five shapes are:

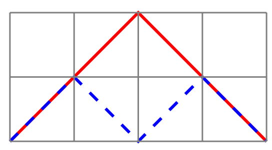


These five configurations exhaust all binary trees with three vertices, illustrating that their count equals $C_3 = 5$.

Proposition 4.6 (Balanced Paths)**.**
The number of paths from $(0, 0)$ to $(2n, 0)$ using steps $(1, 1)$ ("up") and $(1, -1)$ ("down") that never dip below the x-axis is C_n.

Example $(n = 2)$**:** Two valid paths:

- `UUDD`: $\nearrow\nearrow\searrow\searrow$
- `UDUD`: $\nearrow\searrow\nearrow\searrow$



4.1.2 Generating Function and Explicit Formula

Theorem 4.7 (Cauchy product)**.**
Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two absolutely convergent series with sums $A = \sum_{n\geq 0} a_n$ and $B = \sum_{n\geq 0} b_n$.
Define the sequence $(c_n)_{n\geq 0}$ by the Cauchy convolution

$$c_n \;=\; \sum_{k=0}^{n} a_k\, b_{n-k} \quad (n \geq 0)$$

and consider the series $\sum_{n=0}^{\infty} c_n$.
Then the series $\sum_{n\geq 0} c_n$ converges absolutely and

$$\sum_{n=0}^{\infty} c_n \;=\; A\,B.$$

Theorem 4.8 (Generating Function for Catalan Numbers)**.**
The ordinary generating function $C(x)$ for Catalan numbers satisfies:

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Theorem 4.9 (Closed Formula for Catalan Numbers).
The n-th Catalan number is given by:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Lemma 4.10 (Generalized Binomial Coefficient).
For $k \geq 1$:

$$\binom{1/2}{k} = \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!}.$$

4.1.3 Combinatorial Expressions for Catalan Numbers

Proposition 4.11 (Alternative Formulas for Catalan Numbers).
The following identities hold for the n-th Catalan number:

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

and

$$C_n = \binom{2n-1}{n} - \binom{2n-1}{n+1}.$$

4.2 Narayana Numbers

Definition 4.12 (Narayana Numbers).
The **Narayana number**, denoted $N(n, k)$, counts:

- Lattice paths from $(0, 0)$ to $(2n, 0)$ using n up-steps $U = (1, 1)$ and n down-steps $D = (1, -1)$, which do not go below the x-axis, and have exactly k peaks. A peak is an up-step immediately followed by a down-step (a UD sequence).

Example.
The Narayana number $N(3, 2) = 3$. This means:

- There are 3 lattice paths from (0,0) to (6,0) using 3 U-steps and 3 D-steps, staying on or above the x-axis, with exactly 2 peaks. The sequences of steps for these paths are:
  1. UUDUDD (peaks after the 2nd U and 4th U)
  2. UDUUDD (peaks after the 1st U and 3rd U)
  3. UUDDUD (peaks after the 2nd U and 5th U)
- There are 3 non-crossing partitions of the set $\{1, 2, 3\}$ into exactly 2 blocks:
  1. {{1,2}, {3}}
  2. {{1,3}, {2}}
  3. {{2,3}, {1}}

The paths are illustrated below.

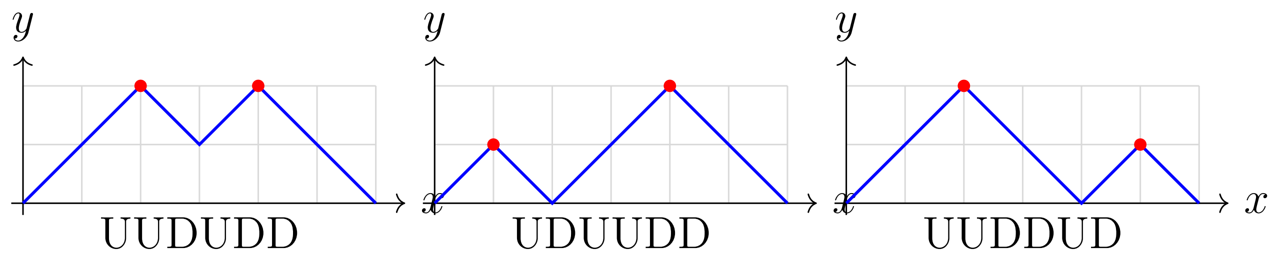


Figure 1: The three lattice paths corresponding to $N(3,2) = 3$. Red dots mark the peaks.

Theorem 4.13 (Binomial Formula for Narayana Numbers).
The Narayana numbers can be calculated using binomial coefficients:

$$N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k} \quad \text{for } 1 \le k \le n.$$

Example (Narayana Table).
The first few Narayana numbers $N(n,k)$:

$N(n,k)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=1$	1	0	0	0	0
$n=2$	1	1	0	0	0
$n=3$	1	3	1	0	0
$n=4$	1	6	6	1	0
$n=5$	1	10	20	10	1

Proposition 4.14 (Connection to Catalan Numbers).
The n-th Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, is the sum of the Narayana numbers in the n-th row:

$$C_n = \sum_{k=1}^{n} N(n,k).$$

4.3 Delannoy Numbers

Definition 4.15 (Delannoy Number).
The **Delannoy number**, denoted $D(m,n)$, counts the number of paths (King's paths) from the origin $(0,0)$ to the point (m,n) in an integer grid using only steps:

- East (E): $(x,y) \to (x+1,y)$
- North (N): $(x,y) \to (x,y+1)$
- Northeast (NE): $(x,y) \to (x+1,y+1)$

Example.
- $D(1,1) = 3$:

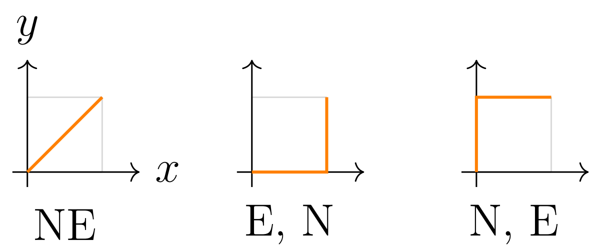


Figure 2: The three paths counted by $D(1,1) = 3$.

- $D(2,1) = 5$:

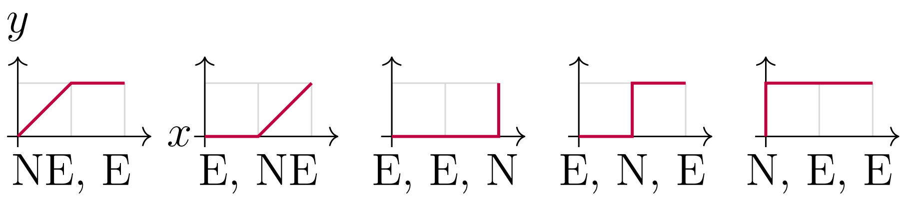


Figure 3: The five paths counted by $D(2,1) = 5$.

- $D(2,2) = 13$:

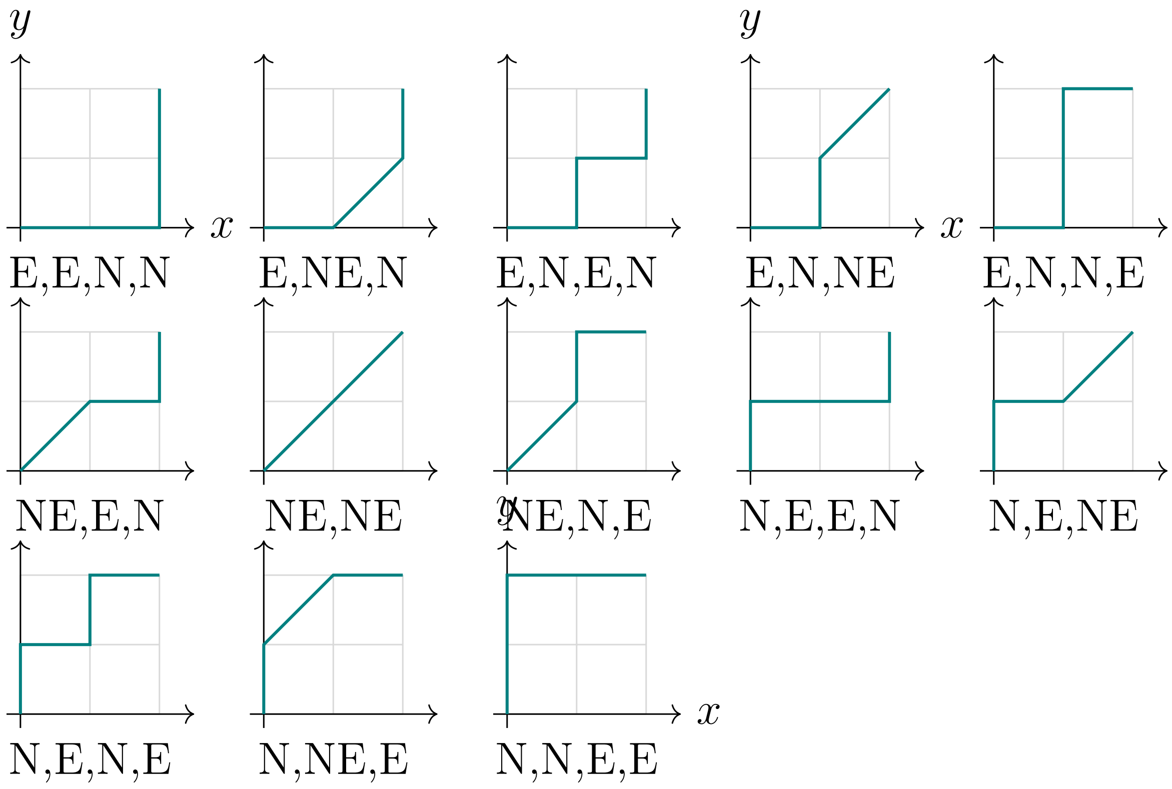


Figure 4: The thirteen paths counted by $D(2,2) = 13$.

Proposition 4.16 (Recurrence Relation for Delannoy Numbers)**.**
For $m, n > 0$, the Delannoy numbers satisfy:

$$D(m,n) = D(m-1,n) + D(m,n-1) + D(m-1,n-1).$$

The boundary conditions are $D(m,0) = 1$ for $m \geq 0$ and $D(0,n) = 1$ for $n \geq 0$. (This implies $D(0,0) = 1$).

Example (Delannoy Table)**.**
The first few Delannoy numbers $D(m,n)$:

$D(m,n)$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$m=0$	1	1	1	1	1
$m=1$	1	3	5	7	9
$m=2$	1	5	13	25	41
$m=3$	1	7	25	63	129
$m=4$	1	9	41	129	321

Proposition 4.17 (Binomial Formulas for Delannoy Numbers)**.**
The Delannoy numbers can also be expressed using binomial coefficients as:

$$D(m,n) = \sum_{j=0}^{\min(m,n)} \binom{m+n-j}{m}\binom{m}{j}$$