Chapter 2: Stirling Numbers and Related Sequences

1. Introduction

This chapter focuses on a fascinating family of such numbers: Stirling numbers of the first and second kind, Lah numbers, and Bell numbers. These numbers arise naturally in problems involving permutations, partitions, and lists.

1.1 Factorial functions

Definition 1.1 (Falling and Rising Factorial Functions).
Falling Factorial: For a non-negative integer n and a real or complex number x, the falling factorial, denoted by $(x)_n$, is defined as:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1)$$

with the convention that $(x)_0 = 1$.
Rising Factorial: For a non-negative integer n and a real or complex number x, the rising factorial (or Pochhammer symbol), denoted by $(x)^{(n)}$, is defined as:

$$(x)^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$$

with the convention that $(x)^{(0)} = 1$.

Remark.
The falling and the rising factorials are also sometimes denoted by $x^{\underline{n}}$ and $x^{\overline{n}}$ respectively.

1.1.1 Properties of Falling and Rising Factorials

- **Relation to factorials**:
  $(n)_n = n!$ (by definition, since $n(n-1)\cdots 1 = n!$), $1^{(n)} = n!$ (rising factorial $1(2)(3)\cdots n = n!$).
- **Symmetry**:
  $$(x)_n = (-1)^n(-x)^{(n)}, \quad x^{(n)} = (-1)^n(-x)_n.$$
  Proof. Factor (-1) from each term in $(x)_n = x(x-1)\cdots(x-n+1)$ to get $(-1)^n(-x)(-x+1)\cdots(-x+n-1) = (-1)^n(-x)^{(n)}$. □
- **Connection to binomial coefficients**: For $x \in \mathbb{C}$,
  $$\binom{x}{k} = \frac{(x)_k}{k!}, \quad \binom{-x}{k} = (-1)^k\frac{x^{(k)}}{k!}.$$
  Proof. The first follows from the binomial and falling factorial's definition; the second uses symmetry $(-x)_k = (-1)^k x^{(k)}$. □
- **Recurrence relations**:
  $$(x)_{n+1} = (x-n)(x)_n, \quad x^{(n+1)} = (x+n)x^{(n)}.$$
  Proof. By definition, appending the next term $(x-n)$ (falling) or $(x+n)$ (rising). □

1.2 Permutations and cycles

A **permutation** π on n elements is a bijective function:

$$\pi : [n] \to [n], \quad \text{where } [n] = \{1, 2, \ldots, n\}.$$

Permutations can be represented in two standard forms:

One-Line Notation

The permutation π is written as a sequence of its outputs:

$$\pi = \pi(1)\ \pi(2)\ \pi(3)\ \cdots\ \pi(n).$$

- The i-th position represents $\pi(i)$.
- Example: For $n = 4$, $\pi = 3\ 1\ 4\ 2$ means:

$$\pi(1) = 3,\ \pi(2) = 1,\ \pi(3) = 4,\ \pi(4) = 2.$$

Two-Line Notation

The permutation π is displayed as a matrix with two rows:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}.$$

- Top row: Domain elements.
- Bottom row: Corresponding images under π.
- Example for $\pi = 3\ 1\ 4\ 2$:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

1.2.1 Cycle Decomposition

Cycle Structure

A permutation decomposes into disjoint **cycles**, which are closed chains of mappings $i \to \pi(i)$. For example, consider the permutation π in one-line notation:

$$\pi = 6\ 5\ 2\ 7\ 1\ 3\ 4\ 8.$$

In two-line notation, this becomes:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 2 & 7 & 1 & 3 & 4 & 8 \end{pmatrix}.$$

Tracing the mappings $i \to \pi(i)$, we observe the following cycles:

$$1 \to 6 \to 3 \to 2 \to 5 \to 1, \qquad 4 \to 7 \to 4, \qquad 8 \to 8.$$

- Each cycle is a closed loop of distinct elements.
- Fixed points (e.g., 8) form 1-cycles .

Cycle Notation

The permutation is written as a product of cycles:

$$\pi = (1\ 6\ 3\ 2\ 5)(4\ 7)(8).$$

Equivalence: Cycles can be reordered or rotated:

$$\pi = (8)(7\ 4)(3\ 2\ 5\ 1\ 6) = (6\ 3\ 2\ 5\ 1)(4\ 7).$$

2. Stirling Numbers

2.1 Stirling Numbers of the First Kind

Definition 2.1 (Stirling numbers of the first kind).
For non-negative integers n and k with $0 \leq k \leq n$, the **Stirling number of the first kind**, denoted $\left[{n \atop k}\right]$, counts:

- The number of ways to seat n people around k non-empty, identical circular tables (rotations of a table do not count as distinct arrangements).
- Equivalently, it counts the number of permutations of n elements that decompose into exactly k disjoint cycles.

Examples.

- For $n = 4$ and $k = 2$, the Stirling number $\left[{4 \atop 2}\right] = 11$ corresponds to:

Cycle Type (3,1) - 8 permutations

$$\begin{array}{cccc}(1\ 2\ 3)(4), & (1\ 3\ 2)(4), & (1\ 2\ 4)(3), & (1\ 4\ 2)(3), \\ (1\ 3\ 4)(2), & (1\ 4\ 3)(2), & (2\ 3\ 4)(1), & (2\ 4\ 3)(1).\end{array}$$

Cycle Type (2,2) - 3 permutations

$$(1\ 2)(3\ 4), \quad (1\ 3)(2\ 4), \quad (1\ 4)(2\ 3).$$

Hence, $\left[{4 \atop 2}\right] = 11$.

- For $n = 3$, the permutations of $[3]$ and their cycle decompositions:
  - $123 \leftrightarrow (1)(2)(3)$ (3 cycles)
  - $132 \leftrightarrow (1)(2\,3)$ (2 cycles)
  - $213 \leftrightarrow (1\,2)(3)$ (2 cycles)
  - $231 \leftrightarrow (1\,2\,3)$ (1 cycle)
  - $312 \leftrightarrow (1\,3\,2)$ (1 cycle)
  - $321 \leftrightarrow (1\,3)(2)$ (2 cycles)

$$\begin{bmatrix}3\\1\end{bmatrix} = 2, \quad \begin{bmatrix}3\\2\end{bmatrix} = 3, \quad \begin{bmatrix}3\\3\end{bmatrix} = 1.$$

This counts: 2 ways to seat at 1 table, 3 ways for 2 tables, 1 way for 3 tables.

2.1.1 Particular Values of Unsigned Stirling Numbers of the First Kind

The unsigned Stirling numbers of the first kind, $\left[{n \atop k}\right]$, satisfy the following special cases:

- $\left[{n \atop 0}\right] = 0$ for $n \geq 1$ (no ways to partition n elements into 0 cycles), and $\left[{0 \atop 0}\right] = 1$ (empty permutation).
- $\left[{n \atop n}\right] = 1$ (each element forms its own 1-cycle).
- $\left[{n \atop 1}\right] = (n-1)!$ (number of cyclic permutations of n elements).
- $\left[{n \atop n-1}\right] = \binom{n}{2}$ (one 2-cycle and $n-2$ fixed points).

2.1.2 Recurrence of $\left[{n \atop k}\right]$

Theorem 2.2 (Recurrence for Stirling Numbers of the First Kind)**.**
For integers $n, k \geq 1$, the following recurrence relation holds:

$$\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n-1\\k-1\end{bmatrix} + (n-1)\begin{bmatrix}n-1\\k\end{bmatrix}$$

with base cases $\left[{n \atop 0}\right] = 0$ for $n \geq 1$, $\left[{0 \atop 0}\right] = 1$, and $\left[{n \atop k}\right] = 0$ if $k > n$.

Combinatorial Proof. Consider seating n people around k non-empty circular tables. Focus on person n. There are two mutually exclusive cases for person n's placement:

1. **Person n sits alone at a new table:** If person n occupies a table by themself, the remaining $n-1$ people must be seated around the other $k-1$ tables. The number of ways to do this is $\left[{n-1 \atop k-1}\right]$.
2. **Person n sits at a table already occupied:** First, arrange the $n-1$ people around k non-empty tables. There are $\left[{n-1 \atop k}\right]$ ways to do this. Now, person n must join one of these k tables. To insert person n into an existing circular arrangement, they can sit to the immediate left of any of the $n-1$ people already seated. Each choice creates a distinct arrangement. Therefore, there are $n-1$ possible positions for person n to join an existing table. The total number of ways for this case is $(n-1)\left[{n-1 \atop k}\right]$.

By the addition principle, summing the possibilities from these two cases gives the total number of ways to seat n people around k tables:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

This completes the proof. □

2.1.3 Triangle of Stirling Numbers of the First Kind

Using the recurrence $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, the triangle is:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

2.2 Stirling Numbers of the Second Kind

Definition 2.3 (Stirling numbers of the second kind)**.**
For non-negative integers n and k with $0 \leq k \leq n$, the **Stirling number of the second kind**, denoted $\left\{{n \atop k}\right\}$ or $S(n,k)$, counts:

- The number of ways to partition a set of n distinct (labelled) elements into k non-empty, indistinguishable (unlabelled) subsets.
- It also counts the number of ways to place n distinct balls into k identical non-empty boxes.

Examples.

- For $n = 4$ and $k = 2$, the Stirling number $\left\{{4 \atop 2}\right\} = 7$ corresponds to the ways to partition the set $\{1,2,3,4\}$ into 2 non-empty subsets:

  Partition Type {3,1} - $\binom{4}{3} = 4$ **partitions**

  $$\{1,2,3\},\{4\}; \quad \{1,2,4\},\{3\}; \quad \{1,3,4\},\{2\}; \quad \{2,3,4\},\{1\}.$$

  Partition Type {2,2} - $\frac{1}{2!}\binom{4}{2} = 3$ **partitions**

  $$\{1,2\},\{3,4\}; \quad \{1,3\},\{2,4\}; \quad \{1,4\},\{2,3\}.$$

  Hence, $\left\{{4 \atop 2}\right\} = 4 + 3 = 7$.
- For $n = 3$, the partitions of $\{1,2,3\}$:
  - Into 1 subset ($k = 1$): $\{1,2,3\}$ (1 way)
  - Into 2 subsets ($k = 2$): $\{1,2\},\{3\}$; $\{1,3\},\{2\}$; $\{2,3\},\{1\}$ (3 ways)
  - Into 3 subsets ($k = 3$): $\{1\},\{2\},\{3\}$ (1 way)

  Thus,

  $$\left\{{3 \atop 1}\right\} = 1, \quad \left\{{3 \atop 2}\right\} = 3, \quad \left\{{3 \atop 3}\right\} = 1.$$

2.2.1 Particular Values of Stirling Numbers of the Second Kind

The Stirling numbers of the second kind, $\left\{{n \atop k}\right\}$, satisfy the following special cases:

- $\left\{ {n \atop 0} \right\} = 0$ for $n \geq 1$ (no ways to partition n elements into 0 non-empty subsets), and $\left\{ {0 \atop 0} \right\} = 1$ (empty set has one partition: into 0 subsets).
- $\left\{ {n \atop n} \right\} = 1$ (each element must be in its own subset).
- $\left\{ {n \atop 1} \right\} = 1$ (all elements must be in the single subset).
- $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$ (one subset must contain exactly two elements, the rest are singletons; choose the pair in $\binom{n}{2}$ ways).
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$ (Each element goes into set A or set B, 2^n ways. Exclude cases where A or B is empty (2 ways). Since sets A, B are indistinguishable, divide by 2: $(2^n - 2)/2 = 2^{n-1} - 1$).

2.2.2 Recurrence of $\left\{ {n \atop k} \right\}$

Theorem 2.4 (Recurrence for Stirling Numbers of the Second Kind)**.**
For integers $n, k \geq 1$, the following recurrence relation holds:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$$

with base cases $\left\{ {n \atop 0} \right\} = 0$ for $n \geq 1$, $\left\{ {0 \atop 0} \right\} = 1$, and $\left\{ {n \atop k} \right\} = 0$ if $k > n$.

Combinatorial Proof. Consider partitioning the set $[n] = \{1, 2, \ldots, n\}$ into k non-empty, unlabelled subsets. Focus on the element n. There are two mutually exclusive cases:

1. **Element n forms a singleton set {n}:** If {n} is one of the partition's subsets, the remaining $n-1$ elements must be partitioned into the other $k-1$ non-empty subsets. The number of ways to do this is $\left\{ {n-1 \atop k-1} \right\}$.
2. **Element n is in a subset with other elements:** First, partition the set $[n-1]$ into k non-empty subsets. There are $\left\{ {n-1 \atop k} \right\}$ ways to do this. Now, element n must join one of these k existing subsets. Since the subsets are formed, they are temporarily distinguishable for the purpose of placing n. There are k choices for which subset element n joins. The total number of ways for this case is $k\left\{ {n-1 \atop k} \right\}$.

By the addition principle, summing the possibilities from these two cases gives the total number of ways to partition $[n]$ into k subsets:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$$

This completes the proof. □

2.2.3 Triangle of Stirling Numbers of the Second Kind

Using the recurrence $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$, the triangle is:

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	1	3	1					
4	0	1	7	6	1				
5	0	1	15	25	10	1			
6	0	1	31	90	65	15	1		
7	0	1	63	301	350	140	21	1	
8	0	1	127	966	1701	1050	266	28	1

2.3 Lah Numbers (Stirling Numbers of the Third Kind)

Definition 2.5 (Lah Numbers)**.**
For non-negative integers n and k with $0 \le k \le n$, the **Lah number**, denoted $\left\lfloor {n \atop k} \right\rfloor$, counts:

- The number of ways to partition n distinct (labeled) elements into k non-empty, indistinguishable (unlabeled) linearly ordered subsets (i.e., sequences where the order within each subset matters).
- Equivalently, it counts the number of ways to arrange n elements into k non-empty lists where the order of elements within each list matters, but the order of the lists themselves does not.

Examples.

- For $n = 3$ and $k = 2$, the Lah number $\left\lfloor {3 \atop 2} \right\rfloor = 6$ corresponds to the ways to partition $\{1, 2, 3\}$ into 2 ordered lists:
$$\{[1], [2, 3]\}, \quad \{[1], [3, 2]\}, \quad \{[2], [1, 3]\}, \quad \{[2], [3, 1]\}, \quad \{[3], [1, 2]\}, \quad \{[3], [2, 1]\}.$$
- For $n = 4$ and $k = 2$, the Lah number $\left\lfloor {4 \atop 2} \right\rfloor = 36$. This counts all partitions into two ordered lists, such as $[1, 2, 3], [4]$ or $[1, 4], [2, 3]$, where internal order matters in each list.
- For $n = 2$, the partitions of $\{1, 2\}$:
  - Into 1 subset ($k = 1$): $[1, 2]$ or $[2, 1]$ (2 ways, but indistinguishable as a single list). However, since the list order matters, $\left\lfloor {2 \atop 1} \right\rfloor = 2$.
  - Into 2 subsets ($k = 2$): $\{[1], [2]\}$ (1 way, as singleton order is fixed).

  Thus,
$$\left\lfloor {2 \atop 1} \right\rfloor = 2, \quad \left\lfloor {2 \atop 2} \right\rfloor = 1.$$

2.3.1 Particular Values of Lah Numbers

The Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ satisfy the following special cases:

- $\left\lfloor {n \atop 0} \right\rfloor = 0$ for $n \ge 1$, and $\left\lfloor {0 \atop 0} \right\rfloor = 1$ (analogous to Stirling numbers).
- $\left\lfloor {n \atop n} \right\rfloor = 1$ (each element forms its own singleton list; order is irrelevant).
- $\left\lfloor {n \atop 1} \right\rfloor = n!$ (all elements form a single ordered list, equivalent to permutations of n elements).
- $\left\lfloor {n \atop n-1} \right\rfloor = \binom{n}{2}$ (one list contains two elements in order, and the remaining $n - 2$ are singletons; choose the pair in $\binom{n}{2}$ ways and order them in 2 ways, but divided by 2 since lists are unordered: $\binom{n}{2} \times 2/2 = \binom{n}{2}$).
- $\left\lfloor {n \atop k} \right\rfloor = 0$ if $k > n$ or $k < 0$.

2.3.2 Recurrence of $\left\lfloor {n \atop k} \right\rfloor$

Theorem 2.6 (Recurrence for Lah Numbers)**.**
For integers $n, k \ge 1$, the following recurrence relation holds:
$$\left\lfloor {n \atop k} \right\rfloor = \left\lfloor {n-1 \atop k-1} \right\rfloor + (n + k - 1)\left\lfloor {n-1 \atop k} \right\rfloor$$
with base cases $\left\lfloor {n \atop 0} \right\rfloor = 0$ for $n \ge 1$, $\left\lfloor {0 \atop 0} \right\rfloor = 1$, and $\left\lfloor {n \atop k} \right\rfloor = 0$ if $k > n$.

Combinatorial Proof. Consider partitioning the set $[n] = \{1, 2, \ldots, n\}$ into k non-empty ordered lists. Focus on the element n. There are two cases:

1. **Element n forms its own list:** The remaining $n - 1$ elements are partitioned into $k - 1$ lists. This contributes $\left\lfloor {n-1 \atop k-1} \right\rfloor$ ways.
2. **Element n joins an existing list:** First, partition the $n - 1$ elements into k lists. There are $\left\lfloor {n-1 \atop k} \right\rfloor$ ways. To insert n into one of these lists, it can be placed:

- At the beginning of any of the k lists.
- Immediately after any of the $n-1$ existing elements in any list.

This gives $k+(n-1)=n+k-1$ insertion positions. Thus, the total for this case is $(n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor$.

Summing these cases gives the recurrence:

$$\left\lfloor {n \atop k} \right\rfloor = \left\lfloor {n-1 \atop k-1} \right\rfloor + (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor$$

□

2.3.3 Triangle of Lah Numbers

Using the recurrence $\left\lfloor {n \atop k} \right\rfloor = (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor + \left\lfloor {n-1 \atop k-1} \right\rfloor$, the triangle is:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	2	1							
3	0	6	6	1						
4	0	24	36	12	1					
5	0	120	240	120	20	1				
6	0	720	1800	1200	300	30	1			
7	0	5040	15120	12600	4200	630	42	1		
8	0	40320	141120	141120	58800	11760	1176	56	1	
9	0	362880	1451520	1693440	846720	211680	28224	2016	72	1

Theorem 2.7.
The Lah numbers, denoted $\left\lfloor {n \atop k} \right\rfloor$, which count the number of ways to partition a set of n elements into k non-empty ordered lists, satisfy the identity:

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1}.$$

Proof. We derive the formula by counting the number of ways to divide n distinct elements into k ordered non-empty lists.

- First, we choose an ordering of the n elements. This can be done in $n!$ ways.
- Then, we must place dividers to split this linear arrangement into k ordered, non-empty blocks (each block forming a list). To form k non-empty parts, we must choose $k-1$ divider positions among the $n-1$ gaps between elements, which can be done in $\binom{n-1}{k-1}$ ways.
- However, since the order of the resulting k lists does not matter in the Lah number definition, we must divide by $k!$ to account for overcounting list permutations.

Therefore, the total number of such partitions is:

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1}.$$

□

2.4 Stirling Numbers, Lah Numbers, and Factorial Functions

Theorem 2.8 (Factorial Expansions)**.**
For non-negative integers n, k $(0 \le k \le n)$ and a real or complex variable x:

$$\textbf{(1)} \quad x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \text{(Stirling numbers of the first kind)},$$

$$\textbf{(2)} \quad x^{n} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k \quad \text{(Stirling numbers of the second kind)},$$

$$\textbf{(3)} \quad x^{(n)} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor (x)_k \quad \text{(Lah numbers)},$$

where:

- Falling factorial: $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$
- Rising factorial: $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$

Proof by Induction. **Proof of (1):** $x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$**.**
Base case $(n = 0)$**:** $x^{(0)} = 1 = \sum_{k=0}^{0} \begin{bmatrix} 0 \\ 0 \end{bmatrix} x^0$.
Inductive step: Assume true for $n-1$:

$$x^{(n-1)} = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k.$$

Multiply by $x + n - 1$:

$$x^{(n)} = (x+n-1) \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k.$$

Expand and reindex:

$$= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k + (n-1) \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k.$$

Combine terms using $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$:

$$x^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \quad \square$$

Proof of (2): $x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k$**.**
Base case $(n = 0)$**:** $x^0 = 1 = \sum_{k=0}^{0} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} (x)_0$.
Inductive step: Assume true for $n-1$:

$$x^{n-1} = \sum_{k=0}^{n-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} (x)_k.$$

Multiply by x:

$$x^n = \sum_{k=0}^{n-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} x(x)_k.$$

Use $x(x)_k = (x)_{k+1} + k(x)_k$:

$$= \sum_{k=0}^{n-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} (x)_{k+1} + \sum_{k=0}^{n-1} k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} (x)_k.$$

Reindex and apply $\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}$:

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k. \quad \square$$

Proof of (3): $x^{(n)} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor (x)_k$**.**
Base case $(n = 0)$**:** $x^{(0)} = 1 = \sum_{k=0}^{0} \left\lfloor {0 \atop 0} \right\rfloor (x)_0$.

Inductive step: Assume true for $n-1$:

$$x^{(n-1)} = \sum_{k=0}^{n-1} \left\lfloor {n-1 \atop k} \right\rfloor (x)_k.$$

Multiply by $x+n-1$:

$$x^{(n)} = (x+n-1)\sum_{k=0}^{n-1} \left\lfloor {n-1 \atop k} \right\rfloor (x)_k.$$

Expand using $(x+n-1)(x)_k = (x)_{k+1} + (n+k-1)(x)_k$:

$$= \sum_{k=0}^{n-1} \left\lfloor {n-1 \atop k} \right\rfloor (x)_{k+1} + \sum_{k=0}^{n-1} (n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor (x)_k.$$

Reindex and apply $\left\lfloor {n \atop k} \right\rfloor = \left\lfloor {n-1 \atop k-1} \right\rfloor + (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor$:

$$x^{(n)} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor (x)_k.$$

□

Examples.

Examples for $n=2$:

- **First Kind Identity:**

$$x^{(2)} = x(x+1) = \begin{bmatrix}2\\1\end{bmatrix} x + \begin{bmatrix}2\\2\end{bmatrix} x^2 = 1\cdot x + 1 \cdot x^2 = x + x^2$$

  Verified: $x(x+1) = x^2 + x$.

- **Second Kind Identity:**

$$x^2 = \begin{Bmatrix}2\\1\end{Bmatrix}(x)_1 + \begin{Bmatrix}2\\2\end{Bmatrix}(x)_2 = 1\cdot x + 1\cdot x(x-1) = x + x^2 - x = x^2$$

  Verified: $x^2 = x^2$.

- **Lah Numbers Identity:**

$$x^{(2)} = \left\lfloor {2 \atop 1} \right\rfloor (x)_1 + \left\lfloor {2 \atop 2} \right\rfloor (x)_2 = 2x + 1\cdot x(x-1) = 2x + x^2 - x = x^2 + x$$

  Verified: $x(x+1) = x^2 + x$.

Examples for $n=3$:

- **First Kind Identity:**

$$x^{(3)} = x(x+1)(x+2) = \begin{bmatrix}3\\1\end{bmatrix} x + \begin{bmatrix}3\\2\end{bmatrix} x^2 + \begin{bmatrix}3\\3\end{bmatrix} x^3 = 2x + 3x^2 + 1x^3 = x^3 + 3x^2 + 2x$$

  Verified: $x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

- **Second Kind Identity:**

$$x^3 = \begin{Bmatrix}3\\1\end{Bmatrix}(x)_1 + \begin{Bmatrix}3\\2\end{Bmatrix}(x)_2 + \begin{Bmatrix}3\\3\end{Bmatrix}(x)_3 = 1\cdot x + 3\cdot x(x-1) + 1\cdot x(x-1)(x-2)$$

  Expand:

$$x + 3(x^2 - x) + (x^3 - 3x^2 + 2x) = x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3$$

  Verified: $x^3 = x^3$.

- **Lah Numbers Identity:**

$$x^{(3)} = \left\lfloor {3 \atop 1} \right\rfloor (x)_1 + \left\lfloor {3 \atop 2} \right\rfloor (x)_2 + \left\lfloor {3 \atop 3} \right\rfloor (x)_3 = 6x + 6\cdot x(x-1) + 1\cdot x(x-1)(x-2)$$

  Expand:

$$6x + 6(x^2 - x) + (x^3 - 3x^2 + 2x) = 6x + 6x^2 - 6x + x^3 - 3x^2 + 2x = x^3 + 3x^2 + 2x$$

  Verified: $x(x+1)(x+2) = x^3 + 3x^2 + 2x$.

Let us denote the Stirling and Lah numbers as follows:

$$s(n,k) := (-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix} \quad \text{(signed Stirling numbers of the first kind)},$$

$$L(n,k) := (-1)^{n-k} \genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k} \quad \text{(signed Lah numbers)},$$
$$S(n,k) := \genfrac{\{}{\}}{0pt}{}{n}{k} \quad \text{(Stirling numbers of the second kind)}.$$

Exercise.
Show the following identities:

(a) $x^n = \sum_{k=0}^{n} (-1)^{n-k} S(n,k) x^{(k)}$,

(b) $(x)_n = \sum_{k=0}^{n} s(n,k) x^k$,

(c) $(x)_n = \sum_{k=0}^{n} L(n,k) x^{(k)}$,

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial and $x^{(k)} = x(x+1)\cdots(x+k-1)$ is the rising factorial.

2.5 Lah Numbers and Stirling Number Identities

Theorem 2.9 (Lah Numbers as a Convolution of Stirling Numbers)**.**
For integers $n \geq k \geq 0$, the Lah numbers satisfy:
$$\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k} = \sum_{j=k}^{n} \genfrac{[}{]}{0pt}{}{n}{j} \genfrac{\{}{\}}{0pt}{}{j}{k}.$$

Proof. We use two factorial expansions from Theorem 2.8:

1. The rising factorial expressed via Lah numbers and falling factorials:
$$x^{(n)} = \sum_{k=0}^{n} \genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k} (x)_k.$$
2. The rising factorial expressed via Stirling numbers of the first kind:
$$x^{(n)} = \sum_{j=0}^{n} \genfrac{[}{]}{0pt}{}{n}{j} x^j.$$

Substitute the falling factorial expansion $x^j = \sum_{k=0}^{j} \genfrac{\{}{\}}{0pt}{}{j}{k} (x)_k$ (from Theorem 2.8(2)) into the second equation:
$$x^{(n)} = \sum_{j=0}^{n} \genfrac{[}{]}{0pt}{}{n}{j} \sum_{k=0}^{j} \genfrac{\{}{\}}{0pt}{}{j}{k} (x)_k.$$
Interchange the order of summation:
$$x^{(n)} = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \genfrac{[}{]}{0pt}{}{n}{j} \genfrac{\{}{\}}{0pt}{}{j}{k} \right) (x)_k.$$
Compare this with the first expansion $x^{(n)} = \sum_{k=0}^{n} \genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k} (x)_k$. Since the falling factorials $(x)_k$ form a basis for polynomials, their coefficients must be equal:
$$\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k} = \sum_{j=k}^{n} \genfrac{[}{]}{0pt}{}{n}{j} \genfrac{\{}{\}}{0pt}{}{j}{k}.$$
$\square$

2.6 Stirling Number Identities

Theorem 2.10 (Orthogonality of Stirling Numbers).
For integers $n \geq m \geq 0$, the unsigned Stirling numbers of the first kind $\left[{n \atop j}\right]$ and the Stirling numbers of the second kind $\left\{{j \atop m}\right\}$ satisfy:

$$\sum_{j=m}^{n} \left[{n \atop j}\right] \left\{{j \atop m}\right\} (-1)^{j-m} = \delta_{nm},$$

$$\sum_{j=m}^{n} \left\{{n \atop j}\right\} \left[{j \atop m}\right] (-1)^{j-m} = \delta_{nm},$$

where δ_{nm} is the Kronecker delta (1 if $n = m$, 0 otherwise).

Exercise.
Prove Theorem 2.10.

Theorem 2.11 (Stirling Number Inversion).
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

(i) $a_n = \sum_{k=0}^{n} \left\{{n \atop k}\right\} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k}\right] a_k$.

(ii) $b_n = \sum_{k=0}^{n} \left[{n \atop k}\right] a_k \Leftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{{n \atop k}\right\} b_k$.

Exercise.
Prove Theorem 2.11.

Theorem 2.12 (Binomial Convolutions for Stirling Numbers).
The Stirling numbers of the first and second kinds satisfy the following convolution identities:

(i) **Horizontal convolution (First Kind):** For integers $n \geq 0$ and $k \geq 0$,

$$\left[{n+1 \atop k+1}\right] = \sum_{j=0}^{n} \left[{n \atop j}\right] \binom{j}{k}.$$

(ii) **Vertical convolution (Second Kind):** For integers $n \geq 0$ and $k \geq 0$,

$$\left\{{n+1 \atop k+1}\right\} = \sum_{j=0}^{n} \binom{n}{j} \left\{{j \atop k}\right\}.$$

Exercise.
Prove Theorem 2.12.

2.7 Generating Functions of Stirling numbers

(For further details on generating functions, please refer to Appendix A at the end of the document.)

Theorem 2.13 (Generating Functions for Stirling Numbers).
For complex numbers t and u:

(i) **Unsigned Stirling numbers of the first kind**:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[{n \atop k}\right] \frac{t^k u^n}{n!} = (1-u)^{-t}.$$

(ii) **Signed Stirling numbers of the first kind**:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} s(n,k) \frac{t^k u^n}{n!} = (1+u)^{t}.$$

(iii) **Stirling numbers of the second kind**:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{{n \atop k}\right\} (t)_k \frac{u^n}{n!} = e^{ut}.$$

Proof.
(i):
We begin with the generating function for the unsigned Stirling numbers of the first kind:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{t^k u^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}t^k\right)\frac{u^n}{n!}\\
&= \sum_{n=0}^{\infty} t^{\overline{n}}\frac{u^n}{n!} \quad \left(\text{using the identity } \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}t^k = t^{\overline{n}}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n(-t)^{\underline{n}}\frac{u^n}{n!} \quad (\text{since } t^{\overline{n}} = (-1)^n(-t)^{\underline{n}})\\
&= \sum_{n=0}^{\infty}\binom{-t}{n}(-u)^n \quad \left(\text{using } \frac{(-t)^{\underline{n}}}{n!} = \binom{-t}{n}\right)\\
&= (1-u)^{-t} \quad (\text{by the binomial theorem}).
\end{aligned}$$

(ii):
The signed Stirling numbers of the first kind, $s(n,k)$, satisfy $s(n,k) = (-1)^{n-k}\left[{n \atop k}\right]$. Substituting this into the generating function:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}s(n,k)\frac{t^k u^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\frac{t^k u^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-t)^k(-u)^n}{n!} \quad (\text{factoring } (-1)^{n-k} = (-1)^n(-1)^{-k})\\
&= (1-(-u))^{-(-t)} \quad (\text{applying part (i) with } t \to -t \text{ and } u \to -u)\\
&= (1+u)^t.
\end{aligned}$$

(iii) Generating function for Stirling numbers of the second kind:
For the Stirling numbers of the second kind, we use the identity $\sum_{k=0}^{n}\left\{{n \atop k}\right\}(t)_k = t^n$:

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}(t)_k\frac{u^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{Bmatrix}n\\k\end{Bmatrix}(t)_k\right)\frac{u^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{t^n u^n}{n!} \quad (\text{by the identity above})\\
&= e^{tu} \quad (\text{recognizing the exponential generating function}).
\end{aligned}$$

□

Theorem 2.14 (Closed Formula for Stirling Numbers of the Second Kind)**.**
The Stirling number $\left\{{n \atop k}\right\}$ is given by:

$$\begin{Bmatrix}n\\k\end{Bmatrix} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

Proof. The number of surjections (onto functions) $f : \{1,\dots,n\} \to \{1,\dots,k\}$ is $k!\left\{{n \atop k}\right\}$. Using inclusion-exclusion we already showed that:

$$Surj(n,k) := k!\begin{Bmatrix}n\\k\end{Bmatrix} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Reindex $j = k - i$, then divide by $k!$:

$$\begin{Bmatrix}n\\k\end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

□

2.8 Exponential Generating Functions for Fixed k

Theorem 2.15 (EGFs for Stirling Numbers)**.**
For integers $k \geq 0$ and complex u:

(i) **Unsigned Stirling numbers of the first kind** ($\begin{bmatrix} n \\ k \end{bmatrix}$):

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^n}{n!} = \frac{(-\ln(1-u))^k}{k!}.$$

(ii) **Signed Stirling numbers of the first kind** ($s(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$):

$$\sum_{n=k}^{\infty} s(n,k) \frac{u^n}{n!} = \frac{(\ln(1+u))^k}{k!}.$$

(iii) **Stirling numbers of the second kind** ($\left\{ n \atop k \right\}$):

$$\sum_{n=k}^{\infty} \left\{ n \atop k \right\} \frac{u^n}{n!} = \frac{(e^u - 1)^k}{k!}.$$

Proof. **Proof of (i):** From the previous Theorem (i):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^k u^n}{n!} = (1-u)^{-t}.$$

Expand $(1-u)^{-t} = e^{-t\ln(1-u)} = \sum_{k=0}^{\infty} \frac{(-t\ln(1-u))^k}{k!}$. Equate coefficients of t^k:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^n}{n!} = \frac{(-\ln(1-u))^k}{k!}. \quad \blacksquare$$

Proof of (ii): From the previous Theorem (ii):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} s(n,k) \frac{t^k u^n}{n!} = (1+u)^t.$$

Expand $(1+u)^t = e^{t\ln(1+u)} = \sum_{k=0}^{\infty} \frac{(t\ln(1+u))^k}{k!}$. Equate coefficients of t^k:

$$\sum_{n=k}^{\infty} s(n,k) \frac{u^n}{n!} = \frac{(\ln(1+u))^k}{k!}. \quad \blacksquare$$

Proof of (iii): Starting from the explicit formula

$$\left\{ n \atop k \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n,$$

consider the exponential generating function:

$$\begin{aligned}
G_k(x) &= \sum_{n=0}^{\infty} \left\{ n \atop k \right\} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \right) \frac{x^n}{n!} \\
&= \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} e^{ix} \\
&= \frac{1}{k!} (e^x - 1)^k.
\end{aligned}$$

The final equality follows from the binomial theorem applied to $(e^x - 1)^k$. $\square$

3. Bell Numbers

Definition 3.1.
The Bell numbers count the number of possible partitions of a set of n elements into non-empty subsets, denoted B_n.

Proposition 3.2.
The n-th Bell number is given by

$$B_n = \sum_{k=0}^{n} \left\{ n \atop k \right\},$$

where $\left\{ n \atop k \right\}$ denotes the Stirling number of the second kind.

Proof. The Stirling number of the second kind $\left\{ n \atop k \right\}$ counts the number of ways to partition a set of n elements into exactly k non-empty subsets. A Bell number B_n counts all such partitions for $k = 1, 2, \ldots, n$. Hence, the formula

$$B_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}$$

holds for all $n \geq 0$. □

Example.
For $n = 3$, consider the set $\{a, b, c\}$. The Bell number B_3 counts these 5 partitions:

1. One subset: $\big\{\{a, b, c\}\big\}$
2. Two subsets:
   - $\big\{\{a\}, \{b, c\}\big\}$
   - $\big\{\{b\}, \{a, c\}\big\}$
   - $\big\{\{c\}, \{a, b\}\big\}$
3. Three subsets: $\big\{\{a\}, \{b\}, \{c\}\big\}$

Thus, $B_3 = 5$. Using the formula:

$$B_3 = \sum_{k=0}^{3} \left\{ 3 \atop k \right\} = \left\{ 3 \atop 1 \right\} + \left\{ 3 \atop 2 \right\} + \left\{ 3 \atop 3 \right\} = 1 + 3 + 1 = 5.$$

Theorem 3.3 (Binomial Convolution for Bell Numbers)**.**
The Bell numbers satisfy:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Proof. By definition, $B_{n+1} = \sum_{j=0}^{n+1} \left\{ n+1 \atop j \right\}$. Using the identity for Stirling numbers of the second kind:

$$\left\{ n+1 \atop j \right\} = \sum_{k=0}^{n} \binom{n}{k} \left\{ k \atop j-1 \right\},$$

we substitute:

$$B_{n+1} = \sum_{j=0}^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left\{ k \atop j-1 \right\}.$$

Interchange summation order and adjust limits (since $\left\{ k \atop j-1 \right\} = 0$ for $j - 1 > k$):

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=1}^{k+1} \left\{ k \atop j-1 \right\}.$$

The inner sum simplifies to the Bell number B_k:

$$\sum_{j=1}^{k+1} \left\{ {k \atop j-1} \right\} = \sum_{m=0}^{k} \left\{ {k \atop m} \right\} = B_k.$$

Thus:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

□

Theorem 3.4 (Exponential Generating Function for Bell Numbers)**.**
The Bell numbers satisfy:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Proof. Recall that $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$. Using the known generating function for Stirling numbers:

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!},$$

sum over all $k \geq 0$:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \right) \frac{x^n}{n!} = \sum_{k=0}^{\infty} \frac{(e^x - 1)^k}{k!} = e^{e^x - 1}.$$

□

Theorem 3.5 (Dobiński's Formula)**.**
For the n-th Bell number B_n,

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Proof. Exercise. □

4. Symmetric polynomial and Stirling numbers

Theorem 4.1.
For integers $n \geq k \geq 1$, the following identities hold:
1. **Unsigned Stirling Numbers of the First Kind**:

$$\left[{n \atop n-k} \right] = \sum_{0 \leq i_1 < i_2 < \cdots < i_k < n} i_1 i_2 \cdots i_k,$$

where the sum runs over all strictly increasing sequences $i_1, i_2, \ldots, i_k$ in $\{0, 1, \ldots, n-1\}$.
2. **Stirling Numbers of the Second Kind**:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{\substack{i_1 + i_2 + \cdots + i_k = n \\ i_j \geq 1}} \frac{n!}{i_1! i_2! \cdots i_k!},$$

where the sum is over all compositions of n into k positive integers i_j.
3. Additionally, the **Unsigned Stirling Numbers of the First Kind** satisfy:

$$\left[{n+1 \atop k+1} \right] = n! \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{i_1 i_2 \cdots i_k}$$

Example.

1. For $n = 4$, $k = 2$:

$$\left[{4 \atop 2} \right] = \sum_{0 \leq i_1 < i_2 < 4} i_1 i_2 = 0 \cdot 1 + 0 \cdot 2 + 0 \cdot 3 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11.$$

2. For $n = 4$, $k = 2$:

$$\left\{ {4 \atop 2} \right\} = \frac{1}{2!} \sum_{\substack{i_1+i_2=4 \\ i_j \geq 1}} \frac{24}{i_1! i_2!} = \frac{1}{2} \left(\frac{24}{1!3!} + \frac{24}{2!2!} + \frac{24}{3!1!} \right) = \frac{1}{2}(4+6+4) = 7.$$

3. For $n = 4$, $k = 2$:

$$\left[{5 \atop 3} \right] = 4! \sum_{1 \leq i_1 < i_2 \leq 4} \frac{1}{i_1 i_2} = 24 \left(\frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} \right).$$

Sum: $\frac{35}{24}$. Multiply: $24 \cdot \frac{35}{24} = 35$.

Appendix A: Ordinary and Exponential Generating Functions

Definition .2 (Generating Function)**.**
The **Generating Function** for an infinite sequence $\{a_n\}_{n=0}^{\infty} = (a_0, a_1, a_2, \dots)$ is the formal power series:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

In this series, the coefficient a_n is the n-th term of the sequence, and the exponent n in x^n indicates its position. This series is often represented by a *compact expression* , and *expanding* this expression generate a power series having the terms of the sequence as coefficients.

Example.
Consider the sequence $(1, 1, 1, 1, \dots)$. Its generating function can be given by the compact expression $\frac{1}{1-x}$. When we expands this expression as a power series, we get:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

The coefficients of this expansion are $1, 1, 1, 1, \dots$, which are exactly the terms of our original sequence.

Definition .3 (Ordinary and Exponential Generating Function)**.**
Given a sequence $(a_n)_{n\geq 0}$,

- The **Ordinary generating function (OGF)** is the formal power series:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

- The **Exponential generating function (EGF)** is the formal power series:

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

Definition .4 (Mixed Generating Function)**.**
Given a double sequence $(a_{n,k})_{n,k\geq 0}$, its **mixed ordinary-exponential generating function** is:

$$A(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k}\, x^n\, \frac{y^k}{k!}.$$

Theorem .5 (Generating Function for Constant and Linear Sequences)**.**
Let $a_n = 1$ for all $n \geq 0$. Then:

$$\text{OGF: } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\text{EGF: } \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Let $a_n = n$. Then:

$$\text{OGF: } \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

$$\text{EGF: } \sum_{n=0}^{\infty} \frac{n x^n}{n!} = x e^x$$

Proof. These results can be verified using the identities from calculus (Taylor series) and algebraic manipulation of formal power series.

For example, we differentiate the geometric series:
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$
Differentiate both sides:
$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$
For EGFs, the Taylor expansion of e^x is well known:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\text{Then } xe^x = \sum_{n=0}^{\infty} \frac{nx^n}{n!}$$
□

Theorem .6 (Generating Function of Fibonacci Numbers)**.**
Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The generating function of the Fibonacci sequence is:
$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

Proof. Let $F(x) = \sum_{n=0}^{\infty} F_n x^n$. We write:
$$F(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots$$
Using the recurrence $F_n = F_{n-1} + F_{n-2}$, we shift:
$$xF(x) = F_0 x + F_1 x^2 + F_2 x^3 + \cdots$$
$$x^2 F(x) = F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots$$
Now subtract:
$$F(x) - xF(x) - x^2 F(x) = F_0 + (F_1 - F_0)x$$
Since $F_0 = 0$, $F_1 = 1$, we get:
$$F(x)(1 - x - x^2) = x \Rightarrow F(x) = \frac{x}{1-x-x^2}$$
□

Remark.
The Fibonacci generating function is a classic example of how recurrence relations can be solved using generating functions without requiring explicit closed-form solutions.