Chapter 1: Solving Recurrence Relations

1. Introduction to Recurrence Relations

Definition 1.1 (Sequence).
sequence is a function
$$a : X \to S,$$
where X is a subset of consecutive integers, typically $\mathbb{N}_0$ or $\mathbb{N}$. The value $a(i)$, denoted by a_i, is called the ith term of the sequence.

- A *finite sequence* has $X = \{i_1, i_2, \dots, i_n\}$ and is written as $(a_{i_1}, a_{i_2}, \dots, a_{i_n})$.
- An *infinite sequence* has $X = \mathbb{N}_0$ or $\mathbb{N}$ and is denoted
$$(a_n)_{n\in\mathbb{N}_0} = (a_0, a_1, a_2, \dots) \quad \text{or} \quad (a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \dots).$$

Definition 1.2 (Recursive Sequence).
A sequence $(a_n)_{n\in\mathbb{N}_0}$ is called a **recursive sequence of order** k if there exists a function $F : S^k \to S$ and initial values $a_0, a_1, \dots, a_{k-1} \in S$ such that
$$a_n = \begin{cases} a_0, a_1, \dots, a_{k-1} \text{ are given as initial values} & \text{if } 0 \le n < k, \\ F(a_{n-1}, a_{n-2}, \dots, a_{n-k}), & \text{if } n \ge k. \end{cases}$$

Definition 1.3 (Linear Recurrence Relation).
A recursive sequence is called a **linear recurrence relation of order** k if it can be written as:
$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} + g(n) \quad \text{for } n \ge k,$$
where $c_1, c_2, \dots, c_k$ are coefficients (which may depend on n) with $c_k \ne 0$, and $g(n)$ is a given function of n.

- If $g(n) = 0$, the recurrence is *homogeneous*.
- If $g(n) \ne 0$, the recurrence is *nonhomogeneous*.

Examples (Recursive Sequences).
The following are fundamental examples of recursively defined sequences:

- **Sequence of Positive Integers:**
$$a_n = \begin{cases} 1, & \text{if } n = 1, \\ a_{n-1} + 1, & \text{if } n \ge 2. \end{cases}$$
- **Arithmetic Progression:** A sequence with initial term a_0 and common difference d:
$$a_n = \begin{cases} a_0, & \text{if } n = 0, \\ a_{n-1} + d, & \text{if } n \ge 1. \end{cases}$$
- **Geometric Progression:** A sequence with initial term a_0 and common ratio r:
$$a_n = \begin{cases} a_0, & \text{if } n = 0, \\ r \cdot a_{n-1}, & \text{if } n \ge 1. \end{cases}$$
- **Fibonacci Sequence:** Defined by
$$F_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-1} + F_{n-2}, & \text{if } n \ge 2. \end{cases}$$
- **Logistic Map:** This recurrence is nonlinear because it involves a product of terms.
$$a_n = r \cdot a_{n-1}(1 - a_{n-1}),$$
where r is a constant.

Definition 1.4 (Closed Form).
A *closed form* for the term a_n is an expression $E(n)$ such that

$$a_n = E(n),$$

that can be evaluated in a finite number of standard operations (addition, multiplication, exponentiation, etc.) with no dependence on prior terms of the sequence.

Remark.
The *order* k indicates the number of preceding terms used to define a_n.

2. Manipulation of Sums

Summations are a fundamental tool in discrete mathematics and recurrence relations. In this subsection, we present several key properties for manipulating sums. These properties allow us to simplify expressions, change summation indices, and rearrange sums to suit our needs.

- **Linearity:**
$$\sum_{k=m}^{n} (c\,a_k + d\,b_k) = c\sum_{k=m}^{n} a_k + d\sum_{k=m}^{n} b_k.$$
- **Splitting:**
$$\sum_{k=m}^{p} a_k + \sum_{k=p+1}^{n} a_k = \sum_{k=m}^{n} a_k.$$
for any $m \le p < n$.
- **Pulling Out Constants:**
$$\sum_{k=m}^{n} c\,a_k = c\sum_{k=m}^{n} a_k.$$
- **Index Shifting:**
$$\sum_{k=m}^{n} a_k = \sum_{j=m+r}^{n+r} a_{j-r}.$$
- **Reversing Order:**
$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} a_{n-k}.$$
- **Sum Swapping (Double Sum):**
$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}.$$
- **Telescoping Sum:** if $a_k = b_{k+1} - b_k$
$$\sum_{k=m}^{n} (b_{k+1} - b_k) = b_{n+1} - b_m.$$

These properties are invaluable for the manipulation of sums, especially when solving recurrence relations and simplifying combinatorial expressions.

2.1 Closed form for elementary finite sums

Examples.

- **Gauss's Sum of Positive Integers**
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. As a 9-year-old, Gauss derived this formula when tasked with summing $1 + 2 + \cdots + 100$. He observed that writing the sum forwards ($S = 1 + 2 + \cdots + n$) and backwards ($S = n + (n-1) + \cdots + 1$) gives $2S = n(n+1)$. Dividing by 2 yields $S = \frac{n(n+1)}{2}$. For odd n, the middle term $\frac{n+1}{2}$ averages the pairs. □

- **Sum of First n Odd Numbers**

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

Proof. Observe $1 = 1^2$, $1+3 = 2^2$, $1+3+5 = 3^2$. Assume true for n: $\sum_{k=1}^{n+1}(2k-1) = n^2 + (2(n+1)-1) = (n+1)^2$. $\square$

- **Sum of First n Even Numbers**

$$\sum_{k=1}^{n} 2k = n(n+1).$$

Proof. Factor 2: $2\sum_{k=1}^{n} k = 2 \cdot \frac{n(n+1)}{2} = n(n+1)$. $\square$

- **Arithmetic Progression Sum** For $a_1, a_1+d, \ldots, a_1+(n-1)d$:

$$S_n = \frac{n}{2}\Big[2a_1 + (n-1)d\Big].$$

Proof. Write S_n forwards and backwards, then add: $2S_n = n[2a_1 + (n-1)d]$. $\square$

- **Geometric Progression Sum** For $a_1, a_1 r, \ldots, a_1 r^{n-1}$ $(r \neq 1)$:

$$S_n = a_1 \frac{1-r^n}{1-r}.$$

Proof. Let $S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1}$. Multiply by r: $rS_n = a_1 r + \cdots + a_1 r^n$. Subtract: $S_n - rS_n = a_1(1-r^n)$. $\square$

- **Sum of Powers** For $p \in \mathbb{N}$, closed forms for $\sum_{k=1}^{n} k^p$ exist. Examples:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Proof. Use induction or Faulhaber's formula. For $p = 2$, verify base case $n = 1$ and assume true for n; show for $n+1$ using algebraic expansion. $\square$

Lemma 2.1.
For any integer $n \geq 1$ and any real numbers x and y, the following factorization holds:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1} = (x-y)\sum_{k=1}^{n} x^{k-1} y^{n-k}.$$

Proof. The proof follows by factoring $x^n - y^n$ as a difference of nth powers. One may verify by expanding the right-hand side or by using induction on n. $\square$

3. First-Order Recurrence Relations

Consider the first-order linear recurrence relation:

$$U_n = \begin{cases} U_0, & \text{if } n = 0 \\ c_n U_{n-1} + g(n), & \text{for } n \geq 1 \end{cases}$$

where $g(n)$ is a given function.

Theorem 3.1.
The closed-form expression for U_n is given by:

$$U_n = \left(\prod_{j=1}^{n} c_j\right) U_0 + \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n} c_j\right) g(i).$$

Proof. We use the substitution method. We write down the terms $U_n, \ldots, U_1$ and multiply the i-th line by the coefficient of U_{n-i-1}. We get:

$$
\begin{array}{llll}
U_n & = & c_n U_{n-1} & + g(n) \\
c_n U_{n-1} & = & c_n c_{n-1} U_{n-2} & + c_n g(n-1) \\
c_n c_{n-1} U_{n-2} & = & c_n c_{n-1} c_{n-2} U_{n-3} & + c_n c_{n-1} g(n-2) \\
c_n c_{n-1} c_{n-2} U_{n-3} & = & c_n c_{n-1} c_{n-2} c_{n-3} U_{n-4} & + c_n c_{n-1} c_{n-2} g(n-3) \\
\vdots & \vdots & \vdots & \vdots \\
\left(\prod_{j=4}^{n} c_j\right) U_3 & = & \left(\prod_{j=3}^{n} c_j\right) U_2 & + \left(\prod_{j=4}^{n} c_j\right) g(3) \\
\left(\prod_{j=3}^{n} c_j\right) U_2 & = & \left(\prod_{j=2}^{n} c_j\right) U_1 & + \left(\prod_{j=3}^{n} c_j\right) g(2) \\
\left(\prod_{j=2}^{n} c_j\right) U_1 & = & \left(\prod_{j=1}^{n} c_j\right) U_0 & + \left(\prod_{j=2}^{n} c_j\right) g(1)
\end{array}
$$

If we sum all the lines we get

$$
\begin{aligned}
U_n &= \left(\prod_{j=1}^{n} c_j\right) U_0 + \sum_{i=1}^{n-1} \left(\prod_{j=i+1}^{n} c_j\right) g(i) + g(n) \\
&= \left(\prod_{j=1}^{n} c_j\right) U_0 + \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} c_j\right) g(i).
\end{aligned}
$$

□

Corollary 3.2.
Let $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_m$ be distinct complex numbers, and let $r \in \mathbb{Z}$. The general term of the linear recurrence sequence $(U_n)_{n \geq 0}$ defined by

$$
U_n = \begin{cases} U_0, & \text{if } n = 0, \\ aU_{n-1} + \sum_{k=1}^{m} x_k y_k^{n+r}, & \text{for } n \geq 1, \end{cases}
$$

has the closed form

$$
U_n = \begin{cases} a^n U_0 + \sum_{k=1}^{m} x_k y_k^{r+1} \left(\dfrac{a^n - y_k^n}{a - y_k}\right), & \text{if } \forall k,\ y_k \neq a, \\ a^n U_0 + \sum_{k=1, k \neq k_0}^{m} x_k y_k^{r+1} \left(\dfrac{a^n - y_k^n}{a - y_k}\right) + n x_{k_0} a^{n+r}, & \text{if } \exists k_0,\ y_{k_0} = a. \end{cases}
$$

Proof. Let $c_i = a$ and $g(n) = \sum_{k=1}^{m} x_k y_k^{n+r}$. By applying Theorem 3.1, we obtain

$$
\begin{aligned}
U_n &= \left(\prod_{j=1}^{n} a\right) U_0 + \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} a\right) \sum_{k=1}^{m} x_k y_k^{n+r} \\
&= a^n U_0 + \sum_{i=1}^{n} a^{n-i} \sum_{k=1}^{m} x_k y_k^{i+r} \\
&= a^n U_0 + \sum_{k=1}^{m} x_k y_k^{r+1} \sum_{i=1}^{n} a^{n-i} y_k^{i-1}.
\end{aligned}
$$

Thus, we have

$$U_n = \begin{cases} a^n U_0 + \sum_{k=1}^{m} x_k y_k^{r+1} \left(\dfrac{a^n - y_k^n}{a - y_k} \right), & \text{if } \forall k,\ y_k \neq a, \\ a^n U_0 + \sum_{k=1, k \neq k_0}^{m} x_k y_k^{r+1} \left(\dfrac{a^n - y_k^n}{a - y_k} \right) + n x_{k_0} a^{n+r}, & \text{if } \exists k_0,\ y_{k_0} = a. \end{cases}$$

□

Corollary 3.3.
The general term of the linear recurrence sequence $(U_n)_{n \geq 0}$ defined by

$$U_n = \begin{cases} U_0, & \text{if } n = 0, \\ aU_{n-1} + b, & \text{for } n \geq 1, \end{cases}$$

has the closed form

$$U_n = \begin{cases} a^n U_0 + b \left(\frac{a^n - 1}{a - 1} \right), & \text{if } a \neq 1 \\ a^n U_0 + nb, & \text{if } a = 1. \end{cases}$$

Proof. Let $c_i = a$ and $g(n) = b$. By applying Theorem 3.1, we obtain

$$\begin{aligned} U_n &= \left(\prod_{j=1}^{n} a \right) U_0 + \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} a \right) b \\ &= a^n U_0 + b \sum_{i=1}^{n} a^{n-i} \\ &= a^n U_0 + b \sum_{i=0}^{n-1} a^i \end{aligned}$$

Therefore

$$U_n = \begin{cases} a^n U_0 + b \left(\frac{a^n - 1}{a - 1} \right), & \text{if } a \neq 1 \\ a^n U_0 + nb, & \text{if } a = 1. \end{cases}$$

□

Exercise.
$U_n = \frac{2}{3} U_{n-1} + 1$ with $U_0 = 0$. Find the closed formula for U_n.

Solution.
Here, $a = \frac{2}{3}$ and $b = 1$. Using the corollary, since $a \neq 1$:

$$U_n = \left(\frac{2}{3} \right)^n (0) + \left(\frac{\left(\frac{2}{3} \right)^n - 1}{\frac{2}{3} - 1} \right) = -3 \left(\left(\frac{2}{3} \right)^n - 1 \right)$$

Thus, $U_n = -3 \left(\left(\frac{2}{3} \right)^n - 1 \right)$.

Remark.
Any closed-form solution derived from solving a recurrence relation should be rigorously verified using mathematical induction. This ensures that the formula holds for all values within its domain.

3.1 Example: Tower of Hanoi

The Tower of Hanoi is a mathematical puzzle involving three pegs (A, B, and C) and a set of disks of different sizes. Initially, all disks are stacked on peg A in decreasing order. The goal is to move the stack to peg C while following these rules:

1. Only one disk may be moved at a time.
2. A larger disk cannot be placed on a smaller one.
3. Disks may only be moved between the three pegs.

Let T_n be the minimum number of moves required to solve the Tower of Hanoi puzzle with n disks. Clearly, we have:

- $T_1 = 1$ (Move the single disk from peg A to peg C).
- $T_2 = 3$ (Move the smaller disk to peg B, the larger disk to peg C, and then the smaller disk to peg C).
- $T_3 = 7$

To solve the Tower of Hanoi problem for n disks, we follow these steps:

1. Move the top $n-1$ disks from peg A to peg B (requiring T_{n-1} moves).
2. Move the largest disk directly from peg A to peg C (1 move).
3. Move the $n-1$ disks from peg B to peg C (requiring another T_{n-1} moves).

This gives the recurrence relation:

$$T_n = 2T_{n-1} + 1, \quad T_0 = 0.$$

Solving this recurrence using the closed-form formula for first-order recurrences:

$$T_n = 2^n - 1.$$

Thus, the closed-form solution for the Tower of Hanoi puzzle is $T_n = 2^n - 1$.

4. Second-Order Recurrence Relations

A second-order recurrence relation expresses each term in terms of the two preceding terms. It is given by:

$$U_n = aU_{n-1} + bU_{n-2} + g(n), \quad \text{for } n \geq 2,$$

where a and b are constants, and $g(n)$ is a given function. The initial values U_0 and U_1 are given.

4.1 Characteristic Equation Method

In the previous section, we discussed how to solve first-order recurrence relations. The approach for solving second-order recurrence relations involves transforming them into first-order recurrence relations and then solving them accordingly.

To achieve this, we introduce the following change of variable:

$$V_n = U_n + \alpha U_{n-1}, \tag{1}$$

where V_n satisfies a first-order recurrence relation:

$$V_n = \beta V_{n-1} + d(n). \tag{2}$$

Substituting (1) into (2) , we obtain:

$$U_n + \alpha U_{n-1} = \beta(U_{n-1} + \alpha U_{n-2}) + d(n).$$

Rearranging, we get:

$$U_n = (\beta - \alpha)U_{n-1} + \alpha\beta U_{n-2} + d(n).$$

Comparing this with the standard second-order recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2} + g(n),$$

we conclude that the parameters must satisfy the system:

$$\begin{cases} \beta - \alpha = a, \\ \alpha\beta = b. \end{cases} \tag{3}$$

This system can also be rewritten as:

$$\begin{cases} \beta - \alpha = a, \\ -\alpha\beta = -b. \end{cases} \tag{4}$$

Lemma 4.1.
Let x and y satisfy the system:

$$\begin{cases} x + y = a, \\ xy = b. \end{cases} \tag{5}$$

Then, x and y are the roots of the quadratic equation:

$$t^2 - at + b = 0. \tag{6}$$

Proof. Consider the quadratic equation:

$$(t - x)(t - y) = 0.$$

Expanding the left-hand side, we obtain:

$$t^2 - (x + y)t + xy = 0.$$

Substituting the given values $x + y = a$ and $xy = b$ from (5), we get:

$$t^2 - at + b = 0.$$

This shows that x and y are indeed the roots of the quadratic equation (6), completing the proof. □

Thus, solving (24) reduces to finding the roots of the characteristic equation:

$$(t + \alpha)(t - \beta) = t^2 - (-\alpha + \beta)t - \alpha\beta = t^2 - at - b = 0. \tag{7}$$

Once we determine α and β, we return to (2), which gives the first-order recurrence:

$$V_n = \beta V_{n-1} + d(n).$$

Using standard techniques, we solve this recurrence to find its general solution, denoted as $h(n)$. Substituting back into (1), we obtain:

$$U_n = -\alpha U_{n-1} + h(n).$$

Finally, solving this first-order recurrence provides the explicit expression for U_n.

Remark.
As with all recurrence relations, any closed-form solution obtained must be verified by mathematical induction.

4.1.1 Example: Fibonacci Sequence

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2, \tag{8}$$

with initial conditions:

$$F_0 = 0, \quad F_1 = 1. \tag{9}$$

To transform this second-order recurrence into a first-order recurrence, we introduce the change of variable:

$$V_n = F_n + \alpha F_{n-1}. \tag{10}$$

where V_n satisfies a first-order recurrence relation:

$$V_n = \beta V_{n-1}. \tag{11}$$

Substituting (10) into (11), we obtain:

$$F_n + \alpha F_{n-1} = \beta(F_{n-1} + \alpha F_{n-2}),$$

which simplifies to:

$$F_n = (\beta - \alpha)F_{n-1} + \alpha\beta F_{n-2}.$$

Comparing with (8), we conclude that the parameters satisfy:

$$\begin{cases} \beta - \alpha = 1, \\ \alpha\beta = 1. \end{cases} \tag{12}$$

Rewriting the system:

$$\begin{cases} \beta - \alpha = 1, \\ -\alpha\beta = -1. \end{cases} \tag{13}$$

Solving for α and β, we derive the characteristic polynomial:

$$(t+\alpha)(t-\beta) = t^2 - (-\alpha+\beta)t - \alpha\beta = t^2 - t - 1 = 0. \tag{14}$$

Applying the quadratic formula:

$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.$$

Thus, the characteristic roots are:

$$-\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}. \tag{15}$$

Remark.
The quantity

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is called the **Golden Ratio**. It plays an important role in mathematics, nature, and various applications in art and architecture. The second root,

$$\bar{\varphi} = \frac{1-\sqrt{5}}{2},$$

is often referred to as the *conjugate* of the Golden Ratio.

Remark.
The choice of assigning $-\alpha$ and β as the first and second solutions, or vice versa, does not affect the final result. This is because the solution structure remains the same, and any change in ordering is absorbed by the constants determined by initial conditions.

Since we obtained α and β, we now solve the first-order recurrence:

$$V_n = \beta V_{n-1}.$$

Using standard techniques, we solve this recurrence to find its general solution:

$$V_n = \beta^{n-1} V_1. \tag{16}$$

From (10), we set $V_1 = F_1 = 1$, so:

$$V_n = \beta^{n-1}, \quad \text{where } \beta = \frac{1-\sqrt{5}}{2}.$$

Substituting back into (10), we obtain:

$$\begin{aligned}
F_n &= -\alpha F_{n-1} + \beta^{n-1}, \\
-\alpha F_{n-1} &= (-\alpha)^2 F_{n-2} + -\alpha\beta^{n-2}, \\
(-\alpha)^2 F_{n-2} &= (-\alpha)^3 F_{n-3} + (-\alpha)^2 \beta^{n-3}, \\
\vdots &= \vdots + \vdots, \\
(-\alpha)^{n-2} F_2 &= (-\alpha)^{n-1} F_1 + (-\alpha)^{n-2}\beta^1, \\
(-\alpha)^{n-1} F_1 &= (-\alpha)^n F_0 + (-\alpha)^{n-1}.
\end{aligned}$$

Summing all the equations, we obtain the explicit formula for the Fibonacci sequence (Binet's formula):

$$\begin{aligned}
F_n &= \sum_{i=0}^{n-1} (-\alpha)^i \beta^{n-1-i} = \frac{(-\alpha)^n - \beta^n}{(-\alpha) - \beta}, \\
&= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).
\end{aligned}$$

Proof by Induction We now prove by induction that:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \bar{\varphi}^n \right), \tag{17}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ is its conjugate.

Base Cases: For $n = 0$:

$$F_0 = \frac{1}{\sqrt{5}} \left(\varphi^0 - \bar{\varphi}^0 \right) = \frac{1}{\sqrt{5}}(1-1) = 0.$$

For $n = 1$:

$$F_1 = \frac{1}{\sqrt{5}} \left(\varphi^1 - \bar{\varphi}^1 \right) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right).$$

Since $\varphi - \bar{\varphi} = \sqrt{5}$, we obtain:

$$F_1 = \frac{\sqrt{5}}{\sqrt{5}} = 1.$$

Inductive Step: Assume that the formula holds for n and $n-1$:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \bar{\varphi}^n \right).$$

Using the Fibonacci recurrence and properties of φ, we conclude:

$$F_{n+1} = \frac{1}{\sqrt{5}} \left(\varphi^{n+1} - \bar{\varphi}^{n+1} \right).$$

Thus, Binet's formula is proven by induction.

Exercise.

The approach used to solve second-order recurrence relations can be extended to third-order recurrence relations. Consider the recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2} + cU_{n-3},$$

where the initial values U_0, U_1, and U_2 are given.

- Propose a systematic method to solve this recurrence relation.
- Derive the associated characteristic polynomial.

4.2 Some properties of Second-Order Recurrence Relation

4.2.1 Binet's Formula

Binet's formula is simply the closed-form expression for any second-order recurrence sequence. For the standard Fibonacci sequence, it is given by:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \bar{\varphi}^n \right), \tag{18}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \quad \bar{\varphi} = \frac{1-\sqrt{5}}{2}$$

are the two roots of the characteristic equation $t^2 - t - 1 = 0$.

Binet's Formula for Homogeneous Second-Order Linear Recurrence Sequence:

We now extend the same method to solve the general homogeneous second-order linear recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}, \quad \text{for } n \geq 2, \tag{19}$$

where the initial values U_0 and U_1 are given.

We consider the general second-order linear recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}, \quad \text{for } n \geq 2, \tag{20}$$

with initial conditions:

$$U_0, \quad U_1 \quad \text{given.} \tag{21}$$

Note that in particular, if the initial conditions are $U_0 = 0$ and $U_1 = 1$, this specific sequence is sometimes referred to as the generalized Fibonacci sequence.

To simplify this second-order recurrence into a first-order recurrence, we introduce the change of variable:

$$V_n = U_n + \alpha U_{n-1}, \tag{22}$$

where V_n satisfies the first-order recurrence:

$$V_n = \beta V_{n-1}. \tag{23}$$

Substituting (22) into (23), we have:

$$U_n + \alpha U_{n-1} = \beta(U_{n-1} + \alpha U_{n-2}),$$

which simplifies to:

$$U_n = (\beta - \alpha)U_{n-1} + \alpha\beta U_{n-2}.$$

Comparing with (20), we derive the system of equations:

$$\begin{cases} \beta - \alpha = a, \\ \alpha\beta = b. \end{cases} \tag{24}$$

The characteristic polynomial for the recurrence is:

$$t^2 - at - b = 0. \tag{25}$$

Applying the quadratic formula, we find its characteristic roots:

$$t = \frac{a \pm \sqrt{a^2 + 4b}}{2}.$$

Thus, the characteristic roots are:

$$\beta = \frac{a + \sqrt{a^2 + 4b}}{2}, \quad -\alpha = \frac{a - \sqrt{a^2 + 4b}}{2}. \tag{26}$$

Since we obtained α and β, we now solve the first-order recurrence:

$$V_n = \beta V_{n-1}.$$

Solving this recurrence explicitly:

$$V_n = \beta^{n-1} V_1, \tag{27}$$

where $V_1 = U_1 + \alpha U_0$.

Substituting back into (22), we obtain:

$$\begin{aligned} U_n &= -\alpha U_{n-1} + \beta^{n-1} V_1, \\ -\alpha U_{n-1} &= (-\alpha)^2 U_{n-2} + (-\alpha)\beta^{n-2} V_1, \\ (-\alpha)^2 U_{n-2} &= (-\alpha)^3 U_{n-3} + (-\alpha)^2 \beta^{n-3} V_1, \\ \vdots &= \vdots + \vdots, \\ (-\alpha)^{n-2} U_2 &= (-\alpha)^{n-1} U_1 + (-\alpha)^{n-2} \beta V_1, \\ (-\alpha)^{n-1} U_1 &= (-\alpha)^n U_0 + (-\alpha)^{n-1} V_1. \end{aligned}$$

Summing all equations, we derive the explicit formula:

$$U_n = (-\alpha)^n U_0 + \sum_{i=0}^{n-1} (-\alpha)^i \beta^{n-1-i} V_1$$

Finally, substituting $V_1 = U_1 + \alpha U_0$, we get the explicit form of the solution:

$$U_n = \begin{cases} (-\alpha)^n U_0 + \dfrac{(-\alpha)^n - \beta^n}{-\alpha - \beta} \left(U_1 + \alpha U_0\right), & \text{if } -\alpha \neq \beta. \\ (-\alpha)^n U_0 + n(-\alpha)^{n-1} \left(U_1 + \alpha U_0\right), & \text{if } -\alpha = \beta. \end{cases} \tag{28}$$

where constants $\beta = \frac{a+\sqrt{a^2+4b}}{2}$, and $-\alpha = \frac{a-\sqrt{a^2+4b}}{2}$.

In particular, if $U_0 = 0$ and $U_1 = 1$, we get the Binet formula of the generalized Fibonacci sequence, given by:

$$U_n = \begin{cases} \dfrac{(-\alpha)^n - \beta^n}{-\alpha - \beta}, & \text{if } -\alpha \neq \beta. \\ n(-\alpha)^{n-1}, & \text{if } -\alpha = \beta. \end{cases} \tag{29}$$

4.2.2 Cassini's Identity: Homogeneous Second-Order Linear Recurrence

Consider the homogeneous second-order linear recurrence relation defined by:

$$U_n = aU_{n-1} + bU_{n-2}, \quad U_0, U_1 \text{ given}.$$

Theorem 4.2 (Cassini's Identity for Homogeneous Second-Order Linear Recurrences).
Then, the generalized Cassini identity holds:

$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2U_0 - U_1^2).$$

Proof. Base Case: For $n = 1$, we verify the identity:

$$U_2U_0 - U_1^2 = (-b)^{1-1}(U_2U_0 - U_1^2),$$

which holds trivially.
Inductive Step: Assume the identity holds for $n = k \geq 1$:

$$U_{k+1}U_{k-1} - U_k^2 = (-b)^{k-1}(U_2U_0 - U_1^2).$$

We need to prove the identity for $n = k + 1$:

$$U_{k+2}U_k - U_{k+1}^2 = (-b)^k(U_2U_0 - U_1^2).$$

Using the recurrence relation $U_{k+2} = aU_{k+1} + bU_k$, we have:

$$U_{k+2}U_k - U_{k+1}^2 = (aU_{k+1} + bU_k)U_k - U_{k+1}^2.$$

Expanding and rearranging terms gives:

$$= aU_{k+1}U_k + bU_k^2 - U_{k+1}^2.$$

Since $U_{k+1} = aU_k + bU_{k-1}$, substitute and simplify:

$$= aU_kU_{k+1} + bU_k^2 - (aU_k + bU_{k-1})U_{k+1}.$$

Simplifying carefully, we get:

$$= b(U_k^2 - U_{k+1}U_{k-1}).$$

Using the induction hypothesis, we have:

$$= b(-(-b)^{k-1}(U_2U_0 - U_1^2)) = (-b)^k(U_2U_0 - U_1^2),$$

thus confirming the identity for $n = k + 1$. By induction, the generalized Cassini identity holds for all $n \geq 1$. □

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4.2.3 Matrix Representations

Introduction to matrices

Definition 4.3 (matrix).
A **matrix** is a rectangular arrangement of numbers in rows and columns. A matrix with m rows and n columns is called an $m \times n$ matrix. The numbers inside a matrix are called its **entries**.

A 2×2 **matrix** has two rows and two columns and is written as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are real or complex numbers.

Matrix Addition and Scalar Multiplication

Two matrices can be added only if they have the same dimensions. Given:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

their sum is obtained by adding corresponding elements:

$$A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

For any real number (scalar) k, we define **scalar multiplication** as:

$$kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

Matrix Multiplication

The product of two matrices is defined only if the number of columns in the first matrix matches the number of rows in the second. For two 2×2 matrices:

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Matrix multiplication is **not commutative**, meaning in general:

$$AB \neq BA.$$

Important: When multiplying both sides of a matrix equation, the multiplication must be done on the *same* side. That is, if we have:

$$AX = B,$$

then multiplying by another matrix C must be done consistently:

$$CAX = CB \quad \text{or} \quad AXC = BC.$$

Changing the order may result in a different matrix or an undefined operation.

Identity Matrix

The **identity matrix** is a special matrix that does not change a matrix when multiplied. The 2×2 identity matrix is:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any 2×2 matrix A:

$$AI_2 = I_2A = A.$$

Determinant of a 2×2 Matrix

The **determinant** of a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by:

$$\det(A) = ad - bc.$$

A matrix is **invertible** if and only if $\det(A) \neq 0$.

Inverse of a 2×2 Matrix

If A is invertible ($\det(A) \neq 0$), its inverse is:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The inverse satisfies:

$$AA^{-1} = A^{-1}A = I_2.$$

Matrix Powers

For a square matrix A, matrix powers are defined by repeated multiplication:

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}, \quad A^0 = I_2.$$

Theorem 4.4.
Let U_n be a sequence defined by the homogeneous recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2},$$

where U_0 and U_1 are given initial conditions, and a, b are real or complex numbers ($a, b \in \mathbb{R}$ or $\mathbb{C}$). Then, the sequence U_n satisfies the following matrix recurrence relation:

$$\begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} U_1 \\ U_0 \end{pmatrix}.$$

Proof. To verify the result, we compute the right-hand side:

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} = \begin{pmatrix} aU_n + bU_{n-1} \\ U_n \end{pmatrix}.$$

By the recurrence relation, we have $U_{n+1} = aU_n + bU_{n-1}$, which matches the first entry of the resulting matrix. The second entry is simply U_n, as expected. This confirms the matrix representation of the recurrence relation. □

If we introduce the notation:

$$X_n = \begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$

then the recurrence relation from the theorem can be rewritten in a more compact form as:

$$X_{n+1} = AX_n.$$

By iterating this relation and using substitution, we obtain:

$$X_{n+1} = A^n X_1.$$

This leads to the natural question: **Can we compute the nth power of the matrix A?**

Theorem 4.5.
Let U_n be the sequence satisfying the recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}.$$

If the associated matrix is given by:

$$A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$

then its $(n-1)$th power is given by:

$$A^{n-1} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

Equivalently, we can write:

$$A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Proof. We proceed by induction on n.
Base Case: For $n = 1$, we check that:

$$A^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The right-hand side also satisfies:

$$\begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1} = I_2.$$

Thus, the formula holds for $n = 1$.
Inductive Step: Assume that for some n, we have:

$$A^{n-1} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

We want to show that this holds for $n+1$, i.e., that:

$$A^n = \begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_n \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

Using the recurrence relation, we compute:

$$A^n = A \cdot A^{n-1}.$$

Substituting the inductive hypothesis:

$$A^n = A \cdot \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

Now, we explicitly compute:

$$A \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Multiplying the matrices:

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}$$

gives:

$$\begin{pmatrix} aU_{n+1} + bU_n & aU_n + bU_{n-1} \\ U_{n+1} & U_n \end{pmatrix}.$$

Using the recurrence relation $U_{n+2} = aU_{n+1} + bU_n$ and $U_{n+1} = aU_n + bU_{n-1}$, we rewrite this as:

$$\begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_n \end{pmatrix}.$$

Thus, we obtain:

$$A \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} = \begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_n \end{pmatrix}.$$

Therefore,

$$A^n = \begin{pmatrix} U_{n+2} & U_{n+1} \\ U_{n+1} & U_n \end{pmatrix} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix}^{-1}.$$

By induction, the formula holds for all $n \geq 1$. □

Cassini's Identity via Determinants:

The previous theorem provides a new proof for **Cassini's identity**, which states that for a second-order linear recurrence sequence:

$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2U_0 - U_1^2).$$

Proof. From the theorem, we have the matrix identity:

$$A^{n-1} \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Taking determinants on both sides and using the property that for any two invertible square matrices X and Y:

$$\det(XY) = \det(X)\det(Y),$$

we obtain:

$$\det\left(A^{n-1}\right) \det \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} = \det \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Since A satisfies $\det(A) = -b$, we have:

$$\det\left(A^{n-1}\right) = (-b)^{n-1}.$$

Defining

$$D_n = \det \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}$$

and

$$D_1 = \det \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix},$$

we obtain:

$$(-b)^{n-1} D_1 = D_n.$$

Since by determinant properties:

$$D_n = U_{n+1}U_{n-1} - U_n^2,$$

and

$$D_1 = U_2U_0 - U_1^2,$$

we conclude:

$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2U_0 - U_1^2).$$

This proves Cassini's identity. □

4.3 Some Classical Second-Order Recurrences

We consider some special cases of the recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2},$$

where U_0 and U_1 are given initial conditions, and a, b are complex numbers. These sequences have well-known applications in number theory, combinatorics, and continued fractions.

Definition 4.6 (Fibonacci and Lucas Sequences).
The **Fibonacci sequence** is defined by:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{with } F_0 = 0, \quad F_1 = 1.$$

The **Lucas sequence** follows the same recurrence but with different initial conditions:

$$L_n = L_{n-1} + L_{n-2}, \quad \text{with } L_0 = 2, \quad L_1 = 1.$$

Definition 4.7 (Pell and Pell-Lucas Sequences).
The **Pell sequence** satisfies:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{with } P_0 = 0, \quad P_1 = 1.$$

The **Pell-Lucas sequence** is given by:

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad \text{with } Q_0 = 2, \quad Q_1 = 2.$$

Definition 4.8 (Jacobsthal and Jacobsthal-Lucas Sequences).
The **Jacobsthal sequence** is defined by:

$$J_n = J_{n-1} + 2J_{n-2}, \quad \text{with } J_0 = 0, \quad J_1 = 1.$$

The **Jacobsthal-Lucas sequence** satisfies:

$$j_n = j_{n-1} + 2j_{n-2}, \quad \text{with } j_0 = 2, \quad j_1 = 1.$$

Theorem 4.9.
Each sequence U_n that starts with the initial conditions $U_0 = 0$ and $U_1 = 1$ is related to its corresponding Lucas sequence V_n (which starts with $V_0 = 2$ and $V_1 = a$) by the identity:

$$V_n = U_{n+1} + U_{n-1}, \quad \text{for } n \geq 1.$$

Proof. The proof follows by induction on n.
Base Case: For $n = 1$, we check:

$$V_1 = 1, \quad U_2 = a, \quad U_0 = 0.$$

Since $U_2 + U_0 = a + 0 = a$, and from the recurrence $V_n = aV_{n-1} + bV_{n-2}$, we obtain the correct value.
Inductive Step: Assume the relation holds for n, i.e.,

$$V_n = U_{n+1} + bU_{n-1}.$$

We need to show that:

$$V_{n+1} = U_{n+2} + bU_n.$$

By the recurrence definition of V_n and U_n, we have:

$$V_{n+1} = aV_n + bV_{n-1}.$$

Substituting the induction hypothesis:

$$V_{n+1} = a(U_{n+1} + bU_{n-1}) + b(U_n + bU_{n-2}).$$

Using the recurrence relation for U_n,

$$U_{n+2} = aU_{n+1} + bU_n, \quad U_n = aU_{n-1} + bU_{n-2},$$

we obtain:

$$V_{n+1} = U_{n+2} + bU_n.$$

This completes the induction, proving the identity. □

Exercise.

For each of the sequences defined above:

1. **Binet's Formula:** Derive the explicit formula for U_n using characteristic roots.
2. **Matrix Representation:** Show that U_n can be written in terms of matrix exponentiation:
$$\begin{pmatrix} U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} U_1 \\ U_0 \end{pmatrix}.$$
3. **Cassini's Identity:** Prove that each sequence satisfies the determinant identity:
$$U_{n+1}U_{n-1} - U_n^2 = (-b)^{n-1}(U_2U_0 - U_1^2).$$

Theorem 4.10.
Let U_n be a sequence satisfying the recurrence relation:

$$U_n = aU_{n-1} + bU_{n-2}, \quad \text{with initial conditions } U_0 = 0, \quad U_1 = 1.$$

Define another sequence V_n by:

$$V_n = U_{n+1} + bU_{n-1}, \quad \text{with } V_0 = 2 \text{ and } V_1 = a.$$

Then, the explicit formulas for U_n and V_n (Binet's formulas) are given by:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation:

$$t^2 - at - b = 0.$$

Proof. The proof follows from the standard method of solving second-order linear recurrence relations. □

Examples.
- Fibonacci and Lucas

4.4 Tiling Problem

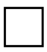
4.4.1 Linear Case

Consider a tiling problem involving a rectangular board of dimensions $1 \times n$, which consists of n labeled cells:

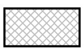
$1 \times n$ board with labeled cells

Our objective is to determine the total number of ways to completely tile this board using the following two types of tiles:

- Squares of size 1×1:



- Dominoes of size 1×2:



This problem explores how different combinations of these tiles can fully cover the board, forming the basis for various combinatorial tiling strategies.

Theorem 4.11.
The number of ways to tile a $1 \times n$ board using identical squares and identical dominoes is given by F_{n+1}, where F_n is the Fibonacci sequence.

Proof. Let U_n denote the number of ways to tile a $1 \times n$ board using identical 1×1 squares and identical 1×2 dominoes.

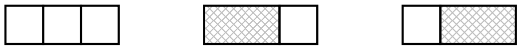
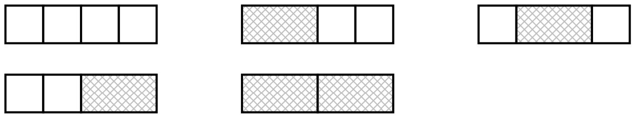
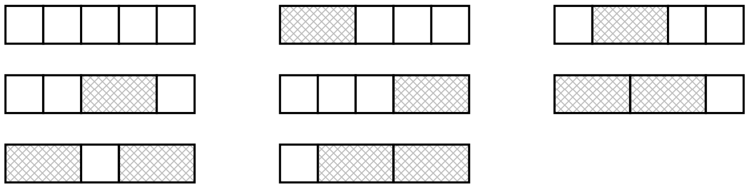
n	**Possible Tilings**
n=1	
n=2	
n=3	
n=4	
n=5	

Table 1: Base Cases for the Linear Tiling Problem

Recursive Case: Consider the last tile placed on a $1 \times n$ board:

- If the last tile is a 1×1 square, the remaining part of the board is a $1 \times (n-1)$ board, which can be tiled in U_{n-1} ways.
- If the last tile is a 1×2 domino, it covers the last two squares, leaving a $1 \times (n-2)$ board, which can be tiled in U_{n-2} ways.

This results in the recurrence relation:

$$U_n = U_{n-1} + U_{n-2}$$

Since this recurrence is identical to that of the Fibonacci sequence, and given the initial conditions $U_1 = 1$ and $U_2 = 2$, it follows that:

$$U_n = F_{n+1}$$

This completes the proof. □

Theorem 4.12.
The Fibonacci sequence satisfies the identity:

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Proof.

- Induction.
- Consider the problem of counting the number of ways to tile a $1 \times n$ board using indistinguishable 1×2 dominoes and 1×1 squares. We have previously established that F_{n+1}, the $(n+1)$-th Fibonacci number, enumerates all possible tilings of the board.
  To analyze these tilings systematically, we classify them based on the number of dominoes k, which may range from 0 (all squares) to $\left\lfloor \frac{n}{2} \right\rfloor$ (the maximum number of dominoes that can fit in the board). Each domino occupies 2 cells, so using k dominoes covers $2k$ cells, leaving $n - 2k$ cells to be filled with squares.
  The problem of arranging k dominoes and $n-2k$ squares is equivalent to solving the equation:
  $$x_1 + x_2 + \cdots + x_{k+1} = n - 2k \quad \text{with} \quad x_i \geq 0,$$
  where x_i represents the number of squares between or adjacent to the dominoes. Specifically:

- x_1: Squares before the first domino
- x_i $(2 \le i \le k)$: Squares between the $(i-1)$-th and i-th domino
- x_{k+1}: Squares after the last domino

By the stars and bars theorem, the number of non-negative integer solutions to this equation is:

$$\binom{(n-2k)+k}{k} = \binom{n-k}{k}.$$

Summing over all valid values of k gives the total number of tilings:

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

The constraint $2k \le n$ ensures $k \le \lfloor \frac{n}{2} \rfloor$, which defines the upper limit of the summation. This combinatorial argument establishes the connection between Fibonacci numbers and domino tilings.

□

4.5 Circular Case

We consider a circular bracelet of length n and investigate the number of ways to tile it using:

- **Identical squares** of size 1×1.
- **Identical dominoes** of size 1×2.

Theorem 4.13.
Let V_n denote the number of ways to tile a circular bracelet of length n using squares and dominoes. Then V_n satisfies the recurrence:

$$V_n = L_n,$$

where L_n is the Lucas number.

Proof. We determine V_n recursively by considering the placement of the last tile.
Base Cases:

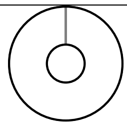
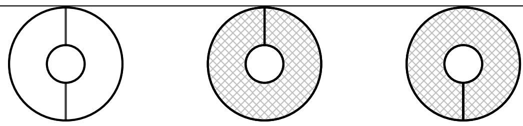
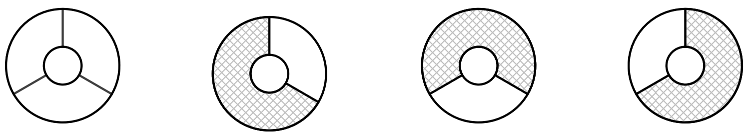
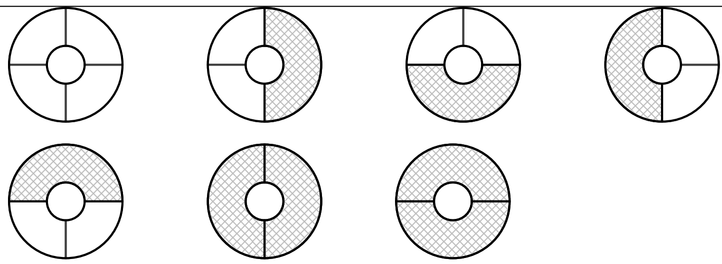
n	**Possible Tilings**
n=1	
n=2	
n=3	
n=4	

Table 2: Base Cases for the Linear Tiling Problem

Recursive Step: Consider a circular bracelet of length n with labeled cells $1, 2, \ldots, n$, where cell n

is adjacent to cell 1. Let V_n denote the number of ways to tile this bracelet using indistinguishable 1×1 squares and 1×2 dominoes. We analyze the tiling possibilities through case decomposition:

- **Case 1: Cell 1 contains a square**
  The remaining $n-1$ cells form a linear chain, which can be tiled in F_n ways, where F_n is the n-th Fibonacci number.
- **Case 2: Cell 1 is covered by a domino**
  There are two distinct circular configurations:
  1. Domino covers cells $(1, 2)$
  2. Domino covers cells $(n, 1)$

  Both configurations leave a linear chain of $n-2$ cells, each tiling in F_{n-1} ways. This contributes $2F_{n-1}$ tilings.

Combining both cases yields the recurrence relation:

$$V_n = F_n + 2F_{n-1}.$$

Using the Fibonacci identity $F_{n+1} = F_n + F_{n-1}$, we restructure the equation:

$$V_n = (F_n + F_{n-1}) + F_{n-1} = F_{n+1} + F_{n-1}.$$

This matches the closed-form expression for the Lucas numbers L_n, which satisfy:

$$L_n = F_{n+1} + F_{n-1}.$$

Through induction on n with base cases:

$$V_1 = 1 = L_1$$
$$V_2 = 3 = L_2$$
$$V_3 = 4 = L_3$$

we establish the equivalence for all $n \geq 1$. Therefore, the number of bracelet tilings corresponds to the Lucas sequence:

$$V_n = L_n.$$

□

Theorem 4.14.
The Lucas sequence satisfies the identity:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

Proof. Consider a circular bracelet of n cells. Let $V_n = L_n$ denote the number of tilings using squares and dominoes. We analyze tilings by considering two cases for cell 1:

- *Case 1: Cell 1 contains a square.* The remaining $n-1$ cells form a linear chain, which can be tiled in $\binom{n-1-k}{k}$ ways with k dominoes.
- *Case 2: Cell 1 is covered by a domino.* There are two configurations:
  - Domino covers cells $(1, 2)$
  - Domino covers cells $(n, 1)$

  Each leaves $n-2$ cells in a linear chain, tiling in $2\binom{n-2-(k-1)}{k-1} = 2\binom{n-1-k}{k-1}$ ways.

Combining both cases:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-1-k}{k} + 2\binom{n-1-k}{k-1} \right]$$

Using the binomial identity $\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1}$, we rewrite:

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-k}{k} + \binom{n-1-k}{k-1} \right]$$

Notice that:

$$\begin{aligned}\frac{n}{n-k}\binom{n-k}{k} &= \binom{n-k}{k} + \frac{k}{n-k}\binom{n-k}{k} \\ &= \binom{n-k}{k} + \binom{n-1-k}{k-1}\end{aligned}$$

Thus:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-k}{k} + \binom{n-1-k}{k-1}\right] = L_n$$

The final equality follows from matching terms with our initial combinatorial expression. This completes the proof. □

4.6 Tiling and generealized sequence of second order

Linear case

Let consider that we want tile the 1‘times n board using squares of a different colors, and dominoes of b different colors,

Theorem 4.15.
The number of ways to tile a $1 \times n$ board using a types of squares and b types of dominoes satisfies both:

1. The linear recurrence:

$$U_n = \begin{cases} 1 & n = 0 \\ a & n = 1 \\ aU_{n-1} + bU_{n-2} & n \geq 2 \end{cases}$$

2. The combinatorial formula:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

Proof.

Recurrence Relation Proof

Consider the last tile in any tiling of a $1 \times n$ board:

- **Case 1: Last tile is a square**
  There are a choices for the square. The remaining $n-1$ cells form a $1 \times (n-1)$ board with U_{n-1} tilings.

  $$\text{Contribution: } aU_{n-1}$$

- **Case 2: Last tile is a domino**
  There are b choices for the domino. The remaining $n-2$ cells form a $1 \times (n-2)$ board with U_{n-2} tilings.

  $$\text{Contribution: } bU_{n-2}$$

Combining both cases gives the recurrence:

$$U_n = aU_{n-1} + bU_{n-2}$$

Combinatorial Proof

Any tiling with k dominoes must contain:

- k dominoes occupying $2k$ cells
- $n-2k$ squares occupying the remaining cells

The number of ways to arrange k dominoes and $n-2k$ squares is equivalent to choosing positions for the dominoes. This is given by the binomial coefficient $\binom{n-k}{k}$, as each domino placement reduces the effective length by k.
Each configuration has:

- a^{n-2k} choices for squares

- b^k choices for dominoes

Summing over all possible k values gives:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

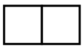
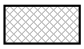
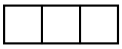
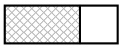
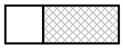
n	Base Case Tilings (a=2, b=3)					
1		(2 ways)				
2		(4 ways)		(3 ways)		
3		(8)		(6)		(6)

Table 3: Example tilings with coefficients $a = 2$, $b = 3$

Both proofs agree through:

$$U_2 = a^2 + b = 2^2 + 3 = 7$$

$$\sum_{k=0}^{1} \binom{2-k}{k} 2^{2-2k} 3^k = \binom{2}{0} 2^2 3^0 + \binom{1}{1} 2^0 3^1 = 4 + 3 = 7$$

This establishes the equivalence between the recursive and combinatorial forms. □

Circular case

Theorem 4.16.
The number of ways to tile a circular bracelet of n labeled cells using a types of squares and b types of dominoes satisfies:

$$V_n = \begin{cases} a & n = 1 \\ U_n + bU_{n-2} & n \geq 2 \end{cases}$$

where U_n is the linear tiling count from Theorem 1. This can alternatively be expressed as:

$$V_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k + b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Proof.

Case Analysis for Circular Tilings

- **Case 1: Cell 1 contains a square**
  The remaining $n-1$ cells form a linear chain (no circular constraint), yielding:

  $$\text{Contribution: } aU_{n-1}$$

- **Case 2: Cell 1 is covered by a domino**
  Two distinct circular configurations exist:
  - Domino covers cells $(1, 2)$
  - Domino covers cells $(n, 1)$

  Each leaves $n-2$ cells in linear arrangement. Total contribution:

  $$\text{Contribution: } 2bU_{n-2}$$

Combining both cases gives the recurrence:

$$V_n = aU_{n-1} + 2bU_{n-2}$$

Combinatorial Interpretation

From Theorem 1, linear tilings are:

$$U_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k$$

Circular tilings include:

- *All linear tilings* (U_n)
- *Wrap-around domino tilings* not counted in linear case:

$$b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Thus:

$$V_n = U_n + bU_{n-2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a^{n-2k} b^k + b \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} a^{n-2-2k} b^k$$

Example Verification

For $n = 3$, $a = 2$, $b = 3$:

$$\begin{aligned}
U_1 &= 2 \\
U_2 &= 2^2 + 3 = 7 \\
U_3 &= 2 \times 7 + 3 \times 2 = 20 \\
V_3 &= 2 \times 7 + 2 \times 3 \times 2 = 14 + 12 = 26 \\
\text{Combinatorial: } &(20) + 3 \times (2) = 26
\end{aligned}$$

□

5. Higher-Order Recurrence Relations

Let's consider the homogeneous linear recurrence relation of order m defined by

$$U_n = \begin{cases} a_1 U_{n-1} + a_2 U_{n-2} + \cdots + a_m U_{n-m}; & n \geq m \\ U_1 = 1; U_0 = U_{-1} = \cdots = U_{-(m-2)} = 0; & -(m-2) \leq n \leq 1 \end{cases}$$

Theorem 5.1.
The $(n+1)$-th term of the homogeneous linear recurrence relation of order m defined above is given by the formula:

$$U_{n+1} = \sum_{k_1 + 2k_2 + \cdots + mk_m = n} \binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}.$$

Proof. Exercise: Induction. □